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SUMMARY

It is well known that the arithmetic mean of two possibly different copulas forms a copula, again. More general, we focus on the weighted power mean (WPM) of two arbitrary copulas which is not necessary a copula again, as different counterexamples reveal. However, various conditions regarding the mean function and the underlying copula are given which guarantee that a proper copula (so-called WPM copula) results. In this case, we also derive dependence properties of WPM copulas and give some brief application to financial return series.

Keywords and phrases: Copulas, generalized power mean, max id, left tail decreasing, tail dependence

1 Introduction

For two given copulas $C_1(u,v)$ and $C_2(u,v)$ we consider the function

$$\overline{C}_r(u,v) \equiv \Big(\alpha C_1(u,v)^r + (1-\alpha)C_2(u,v)^r\Big)^{1/r} \tag{1}$$

where $r \in \mathbb{R}$ and $\alpha \in (0,1)$ on $[0,1]\times[0,1]$. Letting $r \to 0$, (1) reduces to the weighted geometric mean of C_1 and C_2,

$$\overline{C}_0(u,v) = C_1(u,v)^\alpha C_2^{1-\alpha}. \tag{2}$$

Assuming that $\overline{C}_r(u,v)$ is again a copula – which is not guaranteed at all – it is governed by – beside of the specific copula parameters of C_1 and C_2 – by two additional parameters r and α which may allow for more flexibility regarding dependence modelling. In this case, $\overline{C}_r$ will be denoted as *weighted power mean copula*, or briefly WPM copula.

Certainly, there are single results on weighted arithmetic, geometric or harmonic means of two specific copulas (see, e.g. Nelsen, 2006). Nelsen even shows, that certain copula families are closed with respect to the operation in (1), i.e. $\overline{C}_r$

belongs to the same copula family as C_1 and C_2 for a fixed r. But a systematic proof that C is a copula for all r can only be found for the weighted power mean of the maximum- and independence copula (see Fischer & Hinzmann, 2007).

On the other hand, we will provide counterexamples within this paper, where the weighted power mean of two copulas fails to satisfy the postulates of the copula definition. This motivates the derivation of certain criteria for C_1 and C_2 such that $\overline{C}_r$ is again a copula. Therefore, our objective is to give solutions for broad classes of copulas C_1 and C_2 and possibly comprehensive domain of r.

For this reason, the paper is organized as follows: First of all we restrict ourselves to benign copulas C_1 and C_2, where the copula densities exist and, consequently, the two-increasing condition is valid if $\frac{\partial^2 \overline{C}_r}{\partial u \partial v}$ is non-negative. Up to a few special cases (see, for instance, Fischer & Hinzmann, 2007), the direct proof of the two increasing-condition seems to be impossible without assuming the existence of the density. We then show that this sufficient (but not necessary) condition is satisfied for extrem value copulas and positive r. Applying weighted power means with positive r to max-id copulas will also result in proper copulas. As the max-id property of Archimedean copulas can be easily checked, we will provide results for various Archimedean copulas. It will also be shown that combining a copula with a specific positive dependence structure (i.e. left-tail decreasing property) with the independence copula gives again a copula. For $r < 0$, we are only able to derive single results for specific copulas. Afterwards, we investigate how the dependence structure of C_1 and C_2 transforms to the WPM-copula. For the positive quadrant ordering of Lehmann (1966) and the stronger positive ordering of Colangelo (2006), which is based on the concept of tail dependence (so-called LTD-ordering) it will be shown that the WPM-copula (with respect to this orderings) measures a strength/degree of positive ordering which lies between that of C_1 and C_2. This implies that well-known dependence measures like Spearman's ρ and Kendall's τ (which both satisfy this orderings) of WPM-copulas lie also in between those of C_1 and C_2 liegen. However, only numerical approximation to these measures are available because no closed formula can be derived in general. After that, general formulas for the weak and strong tail dependence coefficients of WPM copulas are derived. Finally, we discuss the joint estimation of the WPM parameters r and α together with the copula parameters of C_1 and C_2 and apply the estimation formula to empirical data.

2 A short primer on copulas

Let $[a, b] \subseteq \mathbb{R}$. A function $K : [a, b] \times [a, b] \to \mathbb{R}$ is said to be 2-*increasing* if its K-volume

$$V_K(u_1, u_2, v_1, v_2) \equiv K(u_2, v_2) - K(u_2, v_1) - K(u_1, v_2) + K(u_1, v_1) \geq 0 \tag{3}$$

for all $a \leq u_1 \leq u_2 \leq b$ and $a \leq v_1 \leq v_2 \leq b$. If, additionally, $[a, b] = [0, 1]$ and K satisfies the boundary conditions

$$K(u, 0) = K(0, v) = 0, \quad K(u, 1) = u \text{ and } K(1, v) = v \tag{4}$$

for arbitrary $u, v \in [0, 1]$, K is commonly termed as copula and we write C, instead. For details on copulas we refer to Nelsen, 2006 and Joe, 1999.

Putting a different way, let X and Y denote two random variables with joint distribution $F_{X,Y}(x, y)$ and continuous marginal distribution functions $F_X(x)$ and $F_Y(y)$. According to Sklar's (1959) fundamental theorem, there exists a unique decomposition

$$F_{X,Y}(x, y) = C(F_X(x), F_Y(y))$$

of the joint distribution into its marginal distribution functions and the so-called copula

$$C(u, v) = P(U \leq u, V \leq v), \quad U \equiv F_X(X), \quad V \equiv F_Y(Y) \tag{5}$$

on $[0, 1]^2$ which comprises the information about the underlying dependence structure. From (5) it becomes obvious that a copula is a bivariate distribution function of a pair of random variable (U, V) defined on $[0, 1] \times [0, 1]$. Assuming that C is differentiable with respect to both arguments, equation (3) is satisfied if

$$c(u, v) \equiv \frac{\partial^2 C(u, v)}{\partial u \partial v} \geq 0.$$

Moreover, it is known that

$$\frac{\partial C(u, v)}{\partial u} = P(V \leq v | U = u) \geq 0 \quad \text{and} \quad \frac{\partial C(u, v)}{\partial u} = P(V \leq v | U = u) \geq 0. \tag{6}$$

Later on, we will frequently use the independence copula $C_I(u, v) = uv$ which corresponds to bivariate distributions with independent marginals, the maximum copula $C_U(u, v) = \min\{u, v\}$, associated to random variables which are co-monotone and, thus, constituting an upper bound for all copulas and the minimum copula $C_U(u, v) = \max\{u + v - 1, 0\}$, associated to random variables which are counter-monotone and, thus, constituting an lower bound for all copulas. A copula which comprises minimum, maximum and independence copula is commonly called *comprehensive*.

For a general introduction to copulas we refer to Nelsen (2006) or Joe (1999). A recent overview on the multivariate case is provided by Fischer (2010). Application to finance see Fischer et al. (2009) or Fischer & Köck (2010).

3 Examples and counterexamples

In the literature there are numerous examples of copulas which are specific means of other copulas. Joe (1999), for instance, considers copula B11

$$C(u, v) = \alpha \min(u, v) + (1 - \alpha)uv$$

which composes as weighted arithmetic mean of the maximum copula and of the independence copula. Similarly, copula B12 in Joe (1999)

$$C(u, v) = \min(u, v)^{\alpha}(uv)^{1-\alpha} \quad \alpha \in [0, 1]$$

results as the corresponding geometric mean of both copulas. Nelsen (2006) investigates whether the mean of two copulas with a specific building law is again a copula with the same building law. Consider, for example, the weighted arithmetic mean of two Farlie-Gumbel-Morgenstern copulas with the building law

$$C_i(u, v) = uv + \theta_i u(1 - u)(1 - v), \quad \theta_i \in [-1, 1], \; i = 1, 2,$$

which results again in a Farlie-Gumbel-Morgenstern-Copula:

$$\alpha C_1(u, v) + (1 - \alpha)C_2(u, v) = uv + \alpha(\theta_1 + (1 - \alpha)\theta_2)u(1 - u)v(1 - v).$$

A similar result is proven by Nelsen (2006, p. 107) for weighted geometric means and Gumbel-Barnett copulas which are defined by

$$C_i(u, v) = uv \exp(-\theta_i \ln u \ln v), \quad \theta_i \in (0, 1], \; i = 1, 2.$$

Consequently, the geometric mean of C_1 and C_2,

$$C_1(u, v)^\alpha C_2(u, v)^{1-\alpha} = uv \exp(-(\alpha\theta_1 + (1 - \alpha)\theta_2) \ln u \ln v)$$

is again a Gumbel-Barnett copula. Also the weighted harmonic mean can be successfully used to construct new copulas. Starting from Ali-Mikhail-Haq copulas (see Nelsen, 2006, p. 82) of the form

$$C_i(u, v) = \frac{uv}{1 - \theta_i(1 - u)(1 - v)}, \quad \theta_i \in [-1, 1], \quad i = 1, 2,$$

the weighted harmonic mean

$$\left(\alpha \frac{1}{C_1(u, v)} + (1 - \alpha)\frac{1}{C_2(u, v)}\right)^{-1} = \frac{uv}{1 - (\alpha\theta_1 + (1 - \alpha)\theta_2)(1 - u)(1 - v)}$$

is again a Ali-Mikhail-Haq copula.

Fischer & Hinzmann (2007) turn away from concrete means and consider the broad class of so-called power means which are of the form

$$\overline{C}_r(u, v) = \left(\alpha \min(u, v)^r + (1 - \alpha)(uv)^r\right)^{1/r}, \quad \alpha \in [0, 1]$$

where $r \in \mathbb{R}$. Included as special and limiting cases are the weighted arithmetic ($r = 1$, B11), the weighted geometric ($r \to 0$, B12) and the weighted harmonic mean ($r = -1$). Fischer & Hinzmann (2007) proved that $\overline{C}_r(u, v)$ is again a copula for all $r \in \mathbb{R}$. Due to the simple structure of both copulas, the authors succeed in verifying the 2-increasing-condition even without using the copula density.

In contrast, the question whether in general the weighted power mean of two copulas is again a copula has to be negotiated. Consider, for instance, the following simple counterexample:

Example 1 (Counterexamples) *1. Given the specific power mean of the minimum and the maximum copula, i.e*

$$K(u,v) = (\alpha(uv)^r + (1-a)\max\{u+v-1,0\}^r)^{1/r}$$

with $\alpha = 0.5$ *and* $r = -1$ *(harmonic mean). Plugging* $u_1 = 0.6564$*,* $u_2 = 0.9114$*,* $v_1 = 0.3450$ *and* $v_2 = 0.9240$ *into the formula above,*

$$K(u_2,v_2) = 0.8388,\ K(u_2,v_1) = 0.2825,\ K(u_1,v_2) = 0.5932,\ K(u_1,v_1) = 0.0028.$$

Hence, $K(u_2,v_2) - K(u_1,v_2) - K(u_2,v_1) + K(u_1,v_1) = -0.0341 < 0$ *and the 2-increasing condition is no longer valid, i.e.* $K(u,v)$ *is no copula.*

2. Consider $C(x,y) = \left(0.5\sqrt{C_I(x,y)} + 0.5\sqrt{C_L(x,y)}\right)^2$*, where* C_I *denotes the independence copula and* C_L *the minimum copula. It can be checked that the* C*-volume of* $[0.8,0.9]\times[0.2,0.3]$ *is negative:* $C(0.9,0.3) \approx 0.2336895$*,* $C(0.8,0.3) \approx 0.1624597$*,* $C(0.9,0.2) \approx 0.1370820$*,* $C(0.9,0.3) \approx 0.04$ *and hence*

$$V_C(0.9,0.8,0.3,0.2) = 0.2336895 - 0.1624597 - 0.1370820 + 0.04 \approx -0,0258522 < 0$$

4 Necessary conditions

Obviously, the boundary conditions of a copula are easily checked for the weighted power mean of two given copulas:

$$C(u,1) = (\alpha C_1(u,1)^r + (1-\alpha)C_2(u,1)^r)^{1/r} = (\alpha u^r + (1-\alpha)^r)^{1/r} = u$$

for $0 < u < 1$. The crucial point is the proof of the 2-increasing condition. If we focus on copulas with existing densities, it suffices to show that the second mixed derivative of C is non-negative. In the general case of WPM copula the mixed derivative of C is given by

$$\begin{aligned}\frac{\partial^2}{\partial u \partial v}\overline{C}_r(u,v) &= \frac{\partial^2}{\partial u \partial v}\left(\alpha C_1(u,v)^r + (1-\alpha)C_2(u,v)^r\right)^{\frac{1}{r}} = \\ &= \frac{1}{r}\frac{M^{1/r}}{M^2}\left(\frac{1-r}{r}AB + MC\right),\end{aligned} \tag{7}$$

where $M \equiv \alpha C_1(u,v)^r + (1-\alpha)C_2(u,v)^r \geq 0$,

$$A = \alpha\frac{\partial}{\partial v}C_1(u,v)^r + (1-\alpha)\frac{\partial}{\partial v}C_2(u,v)^r,\ B = \alpha\frac{\partial}{\partial u}C_1(u,v)^r + (1-\alpha)\frac{\partial}{\partial u}C_2(u,v)^r,$$
$$C = \alpha\frac{\partial^2}{\partial u \partial v}C_1(u,v)^r + (1-\alpha)\frac{\partial^2}{\partial u \partial v}C_2(u,v)^r.$$

Whether A, B and C are non-negative depends in particular on the sign and amount of r. For $r \geq 1$ it immediately follows that both A, B (according to equation (6)) and C (due the existence of the copula density of C_i, $i = 1, 2$) are non-negative. On the other hand, the sign of $(1 - r)/rAB$ is non-positive for $r \geq 1$. However, using the equivalent representation

$$\frac{1}{K} \frac{\partial^2 \overline{C}_r(u,v)}{\partial u \partial v} = \alpha^2 \frac{c_1}{C_1} \left(\frac{C_1}{C_2} \right)^r + \alpha(1-\alpha) \frac{c_1}{C_1} + \alpha(1-\alpha) \frac{c_2}{C_2} + (1-\alpha)^2 \frac{c_2}{C_2} \left(\frac{C_2}{C_1} \right)^r$$
$$-(1-r)\alpha(1-\alpha) \left(\frac{\partial \ln C_1}{\partial u} - \frac{\partial \ln C_2}{\partial u} \right) \left(\frac{\partial \ln C_1}{\partial v} - \frac{\partial \ln C_2}{\partial v} \right)$$

$$\text{with} \quad K = \frac{\overline{C}_r(u,v)}{(\alpha C_1(u,v)^r + (1-\alpha) C_2(u,v)^r)^2} C_1(u,v)^r C_2(u,v)^r > 0$$

and where c_i denotes the copula density of C_i for $i = 1, 2$, we derive the sufficient condition

$$\left(\frac{\partial \ln C_1}{\partial u} - \frac{\partial \ln C_2}{\partial u} \right) \left(\frac{\partial \ln C_1}{\partial v} - \frac{\partial \ln C_2}{\partial v} \right) \geq 0 \tag{8}$$

that $\overline{C}_r$ is a copula for all $r \geq 1$. In toto, we proved the very general result for $r \geq 1$:

Theorem 1 *For two copulas C_1, C_2 with*

$$\left(\frac{\partial \ln C_1}{\partial u} - \frac{\partial \ln C_2}{\partial u} \right) \left(\frac{\partial \ln C_1}{\partial v} - \frac{\partial \ln C_2}{\partial v} \right) \geq 0 \tag{9}$$

the weighted power mean of C_1 and C_2 is again a copula for all $r \geq 1$.

Example 2 *Consider the Gumbel-Barnett copulas with different parameters: For $C_i(u,v) = uv \exp(\theta_i \ln u \ln v)$,*

$$\frac{\partial \ln C_i(u,v)}{\partial u} = \frac{1}{u} - \theta_i \frac{\ln v}{u}, \qquad \frac{\partial \ln C_1(u,v)}{\partial u} - \frac{\partial \ln C_2(u,v)}{\partial u} = (\theta_2 - \theta_1) \frac{\ln v}{u} \tag{10}$$

Hence, the expression in brackets in equation (10) are positive for $\theta_1 > \theta_2$ and negative for $\theta_1 < \theta_2$. In both cases, the condition from above is satisfied.

In case of $0 < r < 1$, A and B and hence $(1 - r)/rAB$ are non-negative; the term C may be also negative if the first expression (summand) is negative. For $r < 0$, A and B are non-positive and, consequently, $(1 - r)/rAB$ non-negative. The sign of C is again undetermined, because $r(r - 1) > 0$. To sum up, for $r < 1$ we obtain no general result for copulas C_i, $i = 1, 2$. Additional requirements have to be put on C_i, $i = 1, 2$ to obtain a specific result.

Finally, let's consider the special case of the weighted geometric mean which results for $r \to 0$. For

$$\overline{C}_0(u,v) = C_1(u,v)^\alpha C_2(u,v)^{1-\alpha} = \exp(\alpha \ln C_1(u,v) + (1-\alpha) \ln C_2(u,v)) \tag{11}$$

we derive that

$$\frac{\partial^2}{\partial u \partial v} \exp\left(\alpha \ln C_1(u,v) + (1-\alpha)\ln C_2(u,v)\right) = C(u,v;r=0,\alpha)(DE+F), \quad (12)$$

where

$$D = \alpha \frac{\frac{\partial C_1(u,v)}{\partial u}}{C_1(u,v)} + (1-\alpha)\frac{\frac{\partial C_2(u,v)}{\partial u}}{C_2(u,v)} \geq 0, \quad E = \alpha \frac{\frac{\partial C_1(u,v)}{\partial v}}{C_1(u,v)} + (1-\alpha)\frac{\frac{\partial C_2(u,v)}{\partial v}}{C_2(u,v)} \geq 0$$

$$\text{and} \quad F = \alpha \frac{C_1(u,v)c_1(u,v) - \frac{\partial}{\partial v}C_1(u,v)\frac{\partial}{\partial u}C_1(u,v)}{C_1(u,v)^2} + (1-\alpha)\frac{C_2(u,v)c_2(u,v) - \frac{\partial}{\partial v}C_2(u,v)\frac{\partial}{\partial u}C_2(u,v)}{C_2(u,v)^2}. \quad (13)$$

However, the sign of the expression F is undetermined.

5 Results for specific copula classes

5.1 Extrem-value copulas

Consider now the first restriction that C_i, $i = 1,2$ are extreme-value copulas. In this case $C_i^r(u,v) = C_i(u^r, v^r)$ for $r > 0$ and $i = 1,2$. For expression C we obtain

$$\frac{\partial^2}{\partial u \partial v} C_i^r(u,v) = \frac{\partial^2}{\partial u^r \partial v^r} C_i(u^r, v^r) \frac{\partial}{\partial v} v^r \frac{\partial}{\partial u} u^r \geq 0,$$

such that $AB + MC \geq 0$ holds and we can stated the follow theorem.

Theorem 2 *The weighted power mean $\overline{C}_r(u,v)$ of two arbitrary extrem-value copulas $C_1(u,v)$ and $C_2(u,v)$ is again a copula for $r > 0$.*

Example 3 (WPM-logistic) *Assume that $r > 0$ and $\alpha \in [0,1]$. Combining two possibly different logistic extreme-value copulas by means of a weighted power mean function, the resulting four-parameter copula has the form*

$$C(u,v;\theta) = \left[\alpha e^{\left[(-\ln(u))^{\frac{1}{\lambda_1}} + (-\ln(v))^{\frac{1}{\lambda_1}}\right]^{r\lambda_1}} + (1-\alpha)e^{\left[(-\ln(u))^{\frac{1}{\lambda_2}} + (-\ln(v))^{\frac{1}{\lambda_2}}\right]^{r\lambda_2}}\right]^{\frac{1}{r}}$$

with $\theta \equiv (\alpha, r, \lambda_1, \lambda_2)$ and $0 < \lambda_1, \lambda_2 \leq 1$.

5.2 Max-id copulas

Instead of restricting to extreme-value copulas, we extend our analysis to copulas which are maximal infinitely divisible (max-id). Originally, the max-id property for distribution functions is intensively discussed by Joe (1999). As copulas can be simply extended to distribution functions, we can apply the max-id property direct to copulas. A copula $C(u,v)$ will be denoted as max-id, if $C(u,v)^r$ satisfies the properties of a bivariate distribution function for all $r > 0$. Note that C^r is not a copula for $r \neq 1$ because the boundary conditions are no longer valid. Consider a bivariate copula C with copula density c. In order to be max-id it has to be shown that

$$\frac{\partial^2 C(u,v)^r}{\partial u \partial v} = rC^{r-1}\left(Cc + (r-1)\frac{\partial C}{\partial u}\frac{\partial C}{\partial v}\right) \geq 0 \quad \forall\,(u,v) \in [0,1]\times[0,1]. \qquad (14)$$

Replacing C by $\overline{C}_r$ and after some re-arrangement we obtain

$$\begin{aligned}\frac{1}{K}\frac{\partial^2 \overline{C}_r(u,v)}{\partial u \partial v} &= \alpha^2 \frac{c_1}{C_1}\left(\frac{C_1}{C_2}\right)^r + (1-\alpha)^2 \frac{c_2}{C_2}\left(\frac{C_2}{C_1}\right)^r \\ &+(1-r)\alpha(1-\alpha)\frac{\partial \ln C_1}{\partial v}\frac{\partial \ln C_2}{\partial u} + (1-r)\alpha(1-\alpha)\frac{\partial \ln C_1}{\partial u}\frac{\partial \ln C_2}{\partial v} \\ &+\alpha(1-\alpha)\frac{1}{C_1^2}\left(C_1 c_1 - (1-r)\frac{\partial C_1}{\partial u}\frac{\partial C_1}{\partial v}\right) \\ &+\alpha(1-\alpha)\frac{1}{C_2^2}\left(C_2 c_2 - (1-r)\frac{\partial C_2}{\partial u}\frac{\partial C_2}{\partial v}\right)\end{aligned}$$

$$\text{for } K \equiv \frac{C_m}{(\alpha C_1^m + (1-\alpha)C_2^m)^2} C_1^m C_2^m > 0.$$

This expression is positive, if C_1 and C_2 satisfy condition (14), i.e. are both max-id. Obviously, this assertion also holds for $r = 0$, because $D, E \geq 0$ and the max-id property in (13) guarantees that $F \geq 0$. Hence, we derived the following result:

Theorem 3 *Assume that $C_1(u,v)$ and $C_2(u,v)$ are two max-id copulas and $r \in [0,1)$. Then $\overline{C}_r$ is again a copula.*

We conclude with two examples of max-id copulas.

Example 4 *The logarithm of the Galambos copula with parameter $\delta \in (0,\infty)$ is given by*

$$\log C_\delta = \log u + \log v + \left((-\log u)^{-\delta} + (-\log u)^{-\delta}\right)^{-1/\delta}$$

and hence

$$\frac{\partial^2 \log C_\delta(u,v)}{\partial u \partial v} = (a+\delta)\left((-\log u)^{-\delta} + (-\log v)^{-\delta}\right)^{\frac{-1}{\delta}-2}\left((-\log u)(-\log v)\right)^{-\delta-1} \geq 0.$$

Example 5 *Taking the logarithm of the FGM-Copula $C_\theta(u,v) = uv(1+\theta(1-u)(1-v))$ with parameter $\theta \in [-1,1]$,*

$$\log C_\theta(u,v) = \log u + \log v + \log\left(1+\theta(1-u)(1-v)\right).$$

The corresponding derivatives are given by

$$\frac{\partial^2 \log C_\theta(u,v)}{\partial u \partial v} = \frac{\theta}{\left(1+\theta(1-u)(1-v)\right)^2}, \tag{15}$$

which means that the max-id property depends only from the sign of the parameter.

In particular, Archimedean copula C_ϕ with continuous, convex and strictly monotone decreasing generators ϕ – which guarantees that the inverse function ϕ^{-1} exists – are max-id if and only if $-\ln \phi^{-1} \in \mathcal{L}_2^*$, where

$$\mathcal{L}_2^* \equiv \{\omega : [0,\infty] \to [0,\infty], \omega(\infty) = \infty, \text{sign}(\omega^{(i)}) \neq (-1)^i, i = 1,2\}.$$

The following table summarizes Archimedean copulas for which the max-id property has already been verified (Nikoloulopoulos & Karlis, 2010 and own calculations), the names and equations are adopted from Nelsen (2006).

Name	Copula function	Parameter
Clayton	$\left(u^{-\theta} + v^{-\theta} - 1\right)^{-\frac{1}{\theta}}$	$\theta > 0$
AMH	$\frac{uv}{1-\theta(1-u)(1-v)}$	$\theta \geq 0$
Gumbel	$\exp\left(-\left((-\ln u)^\theta + (-\ln v)^\theta\right)^{\frac{1}{\theta}}\right)$	$\theta \geq 1$
Frank	$-\frac{1}{\theta}\ln\left(1 + \frac{(e^{-\theta u}-1)(e^{-\theta v}-1)}{e^{-\theta}-1}\right)$	$\theta > 0$
Joe	$1 - \left((1-u)^\theta + (1-v)^\theta + (1-u)^\theta(1-v)^\theta\right)^{\frac{1}{\theta}}$	$\theta \geq 1$
(4.2.12)	$\left(1 + \left((u^{-1}-1)^\theta + (v^{-1}-1)^\theta\right)^{\frac{1}{\theta}}\right)^{-1}$	$\theta \geq 1$
(4.2.13)	$\exp\left(1 - \left((1-\ln u)^\theta + (1-\ln v)^\theta - 1\right)^{\frac{1}{\theta}}\right)$	$\theta > 1$
(4.2.14)	$\left(1 + \left((u^{-\frac{1}{\theta}}-1)^\theta + (v^{-\frac{1}{\theta}}-1)^\theta\right)^{\frac{1}{\theta}}\right)^{-\theta}$	$\theta \geq 1$
(4.2.19)	$\frac{\theta}{\ln\left(e^{\frac{\theta}{u}} + e^{\frac{\theta}{v}} - e^\theta\right)}$	$\theta > 0$
(4.2.20)	$\left(\ln\left(\exp(u^{-\theta}) + \exp(v^{-\theta}) - \exp(1)\right)\right)^{-\frac{1}{\theta}}$	$\theta > 0$

Table 1: Selected Archimedean max-id copulas

5.3 Left tail decreasing property and independence copula

Both copulas B11 and B12 in Joe (1999) and the copula discussed in Fischer & Hinzmann (2006) result as a mixture of a copula C_1 with the independence copula

$C_2(u,v) = uv$. For the independence copula we easily conclude that $\frac{\partial \ln(C_2)}{\partial u} = \frac{1}{u}$ and $\frac{\partial \ln(C_2)}{\partial v} = \frac{1}{v}$, respectively. Hence, condition (8) re-writes as

$$\left(\frac{\partial \ln(C_1)}{\partial u} - \frac{1}{u}\right)\left(\frac{\partial \ln(C_1)}{\partial v} - \frac{1}{v}\right) \geq 0. \qquad (16)$$

This condition is closely related to a property of positive dependence: Assume that U and V are random variables with copula C as bivariate distribution function. V given U is called "left-tail decreasing" (briefly: $\text{LTD}_{V|U}$) if $\frac{C(u,v)}{u}$ is non-increasing in u. Exchanging the part of u and v, we say that U given V is "left-tail decreasing". If $\text{LTD}_{V|U}$ and $\text{LTD}_{U|V}$ holds for the underlying copula C, C is briefly denoted as "left tail decreasing" (LTD-) copula. An equivalent condition for the LTD-property of a copula is

$$\frac{\partial \ln C(u,v)}{\partial u} - \frac{1}{u} \leq 0 \text{ und } \frac{\partial \ln C(u,v)}{\partial v} - \frac{1}{v} \leq 0,$$

Hence, for a LTD-copula C_1 we state the following result:

Theorem 4 *The weighted power mean of a LTD-copula and the independence copula is again a copula for $r > 0$.*

The following example illustrates the procedure for a specific Archimedian copula (which is max-id for $\theta > 1$).

Example 6 *Consider the copula (4.2.12) in Nelsen (1999):*

$$C_1(u,v) = \exp\left(1 - \left((1-\ln(u))^\theta + (1-\ln(v))^\theta - 1\right)^{1/\theta}\right), \quad \theta > 0$$

with partial derivatives

$$\frac{\partial \ln C_1}{\partial u} = -\frac{\left((1-\ln(u))^\theta + (1-\ln(v))^\theta - 1\right)^{\frac{1}{\theta}-1}(1-\ln u)^{\theta-1})}{u} < 0, \quad \frac{\partial \ln C_2}{\partial v} < 0.$$

Consequently,

$$\left(\frac{\partial \ln(C_1)}{\partial u} - \frac{1}{u}\right)\left(\frac{\partial \ln(C_1)}{\partial v} - \frac{1}{v}\right) \geq 0.$$

We are also able to derive the result of Fischer & Hinzmann (2007) as a special case because the maximum copula is a LTD-Copula.

Example 7 *For $C_1 = \min(u,v)$ we have*

$$\frac{\partial \ln C_1(u,v)}{\partial u} = \begin{cases} \frac{1}{u} \text{ für } u < v \\ 0 \text{ sonst} \end{cases} \quad \text{and} \quad \frac{\partial \ln C_1(u,v)}{\partial v} = \begin{cases} \frac{1}{v} \text{ für } u > v \\ 0 \text{ sonst} \end{cases}$$

Hence,

$$\left(\frac{\partial \ln \min(u,v)}{\partial u} - \frac{1}{u}\right)\left(\frac{\partial \ln \min(u,v)}{\partial v} - \frac{1}{v}\right) = 0$$

for $u, v \in [0,1]$. This explains why the weighted power mean of the maximum and the independence copula is always a copula (for the direct proof we refer to Fischer & Hinzmann (2007).

Finally, we consider a copula which is not max-id.

Example 8 *The Gumbel-Barnett-Copula is of the form*

$$C_1(u,v) = uve^{-\theta \ln u \ln v} \quad \Longrightarrow \quad \frac{\partial \ln C_1}{\partial u} = \frac{1}{u} - \theta \cdot \frac{\ln v}{u}.$$

I.e. it is a LTD-copula. Therefore,

$$\left(\frac{\partial \ln C_1}{\partial u} - \frac{1}{u}\right)\left(\frac{\partial \ln C_1}{\partial v} - \frac{1}{v}\right) = \theta^2 \frac{\ln u \ln v}{uv} > 0,$$

which implies that the weighted power mean of the Gumbel-Barnett copula and the independence copula is a copula for all $r \geq 1$. Moreover,

$$c_1 = e^{-\theta \ln u \ln v}\left(\theta^2 \ln u \ln - \theta(\ln u + \ln v) + (1-\theta)\right)$$

and

$$\frac{c_1}{C_1} = \frac{\theta^2 \ln u \ln - \theta(\ln u + \ln v) + (1-\theta)}{uv}.$$

For $C_2(u,v) = uv$ we have

$$\left(\frac{C_1}{C_2}\right)^r = e^{-\theta r \ln u \ln v} \quad und \quad \left(\frac{C_2}{C_1}\right)^r = e^{\theta r \ln \ln v}.$$

Note, that for $0 \leq r < 1$

$$\begin{aligned}
\alpha(1-\alpha)\frac{c_1}{C_1} &= \alpha(1-\alpha)\theta^2 \frac{\ln u \ln v}{uv} - \alpha(1-\alpha)\theta \frac{\ln u + \ln v}{uv} \\
&\quad + \alpha(1-\alpha)(1-\theta)\frac{1}{uv} \\
&\geq (1-r)\alpha(1-\alpha)\theta^2 \frac{\ln u \ln v}{uv} \\
&= (1-r)\alpha(1-\alpha)\left(\frac{\partial \ln C_1}{\partial u} - \frac{1}{u}\right)\left(\frac{\partial \ln C_1}{\partial v} - \frac{1}{v}\right),
\end{aligned}$$

such that the weighted power mean of the Gumbel-Barnett-copula und the independence copula is a copula for $0 \leq r < 1$, too.

5.4 Complementary results for $r < 0$

The difficulty for $r < 0$ of the proof that the power mean of two copulas is again a copula will again be illustrated by means of the Gumbel-Barnett copula, where we can prove the result only for a restricted range of the parameter set.

Example 9 *Again, we focus on a Gumbel-Barnett copula* $C_1(u,v) = uve^{-\theta \ln u \ln v}$*. The series representation of the exponential expression reads as*

$$e^{\theta(-r)\theta \ln u \ln v} = \sum_{i=0}^{\infty} \frac{(-r)^i \theta^i (\ln u \ln v)^i}{i!}$$

and contains completely positive addends if $-r > 0$ *and* $\theta > 0$*. Subtracting* $\alpha(1-\alpha)(-r)\frac{\ln u}{\ln v}uv$ *from* $(1-\theta)/(uv)\alpha^2$ *times the first addend of this series representation we obtain* $\alpha(1-\alpha)(-r)\frac{\ln u}{\ln v}uv$*, such that* $(-r)\left(\alpha^2(1-\theta)\theta - \alpha(1-\alpha)\theta^2\right)\ln u \ln v/(uv)$*. This difference is positive if* $\alpha > \theta$*. That means in this special case where either the range of the dependence parameter* θ *is restricted or the range/domain of the weighting factor* α *is restricted, the weighted power mean of a Gumbel-Barnett and the independence copula is again a copula for* $r < 0$*.*

6 Properties of positive dependence and ordering

The LTD-property is only one form of positive dependence of random variables. For a detailed treatment of that topic we refer to Joe (1999) or Nelsen (2006). ¿From these properties we can derive orderings of positive dependence. Probably the most famous one is the positive dependence ordering (briefly: PQD-ordering) of Lehmann (1966). In this context, a copula C_1 is said to be less positive dependent than a copula C_2 (in short: $C_1 \leq_{\text{PQD}} C_2$) if $C_1(u,v) \leq C_2(u,v)$ for all $u, v \in [0,1]$. Similarly, based on the LTD-property, we can introduce a LTD-ordering of positive dependence. According to Colangelo (2006), C_1 has less positive dependence than C_2 (in short: $C_1 \leq_{\text{LTD}} C_2$) if $D(u,v) \equiv (C_1(u,v) - C_2(u,v))/u$ is monotone decreasing in u for all $u, v \in [0,1]$. For differentiable copulas it suffices to demonstrate that $\partial D(u,v)/\partial u \leq 0$ for all $u, v \in [0,1]$. It can be shown that LTD-ordering implies PQD-ordering.

Assume now that the positive dependence of C_1 is weaker than that of C_2 in the sense of one of that orderings. It raises the question whether the positive dependence of the corresponding WPM-copula is stronger than that of C_1 and weaker than that of C_2. The answer turns out to be very simple for the PDQ-ordering due to the fact that every weighted power mean lies in between the minimum and the maximum of all values. Not so simple is the answer for the LTD-ordering as the next theorem shows.

Theorem 5 *For given copulas* C_1 *and* C_2 *with* $C_1 \leq_{LTD} C_2$ *let* $\overline{C}_r$ *denote the corresponding WPM-Copula with parameters* $\alpha \in [0,1]$ *and* $r \in \mathbb{R}$*. Then holds*

$$C_1 \leq_{LTD} \overline{C}_r \text{ und } \overline{C}_r \leq_{LTD} C_2.$$

Proof: Note first that for every copula C holds

$$\frac{\partial C/u}{\partial u} = \frac{C}{u}\left(\frac{\partial \ln C}{\partial u} - \frac{1}{u}\right).$$

After some simple re-formulations we obtain

$$\frac{\partial \ln \overline{C}_r}{\partial u} - \frac{1}{u} = g\left(\frac{\partial \ln C_1}{\partial u} - \frac{1}{u}\right) + (1-g)\left(\frac{\partial \ln C_2}{\partial u} - \frac{1}{u}\right)$$

with weight

$$0 \leq g = \frac{\alpha C_1^r}{\alpha C_1^r + (1-\alpha)C_2^r} \leq 1.$$

Therefore, $\left(\frac{\partial \ln \overline{C}_r}{\partial u} - \frac{1}{u}\right)$ is a weighted arithmetic mean of $\left(\frac{\partial \ln C_1}{\partial u} - \frac{1}{u}\right)$ and $\left(\frac{\partial \ln C_2}{\partial u} - \frac{1}{u}\right)$ which implies that

$$\left(\frac{\partial \ln C_1}{\partial u} - \frac{1}{u}\right) \leq \left(\frac{\partial \ln \overline{C}_r}{\partial u} - \frac{1}{u}\right) \leq \left(\frac{\partial \ln C_2}{\partial u} - \frac{1}{u}\right).$$

It follows from the property of PQD-orderings that $C_1 \leq \overline{C}_r \leq C_2$ and, consequently,

$$\frac{\overline{C}_r}{u}\left(\frac{\partial \ln \overline{C}_r}{\partial u} - \frac{1}{u}\right) \geq \frac{C_1}{u}\left(\frac{\partial \ln C_1}{\partial u} - \frac{1}{u}\right) \quad \text{(i.e. } C_1 \leq_{\text{LTD}} \overline{C}_r)$$

and

$$\frac{\overline{C}_r}{u}\left(\frac{\partial \ln \overline{C}_r}{\partial u} - \frac{1}{u}\right) \leq \frac{C_2}{u}\left(\frac{\partial \ln C_1}{\partial u} - \frac{1}{u}\right) \quad \text{(i.e. } C_2 \geq_{\text{LTD}} \overline{C}_r).$$

7 Measures of dependence

Copula-based dependence measures T are nothing else but specific mappings from the space of bivariate copulas to the interval $[-1,1]$. As an essential requirement to such a mapping we have to proclaim that it preserves elementary dependence orderings. In case of the LTD-ordering we have to make sure that

$$C_1 \leq_{\text{LTD}} C_2 \Longrightarrow T(C_1) \leq T(C_2).$$

In this case, the domain of T for the WPM-copula $\overline{C}_r$ is determined by $[T(C_1), T(C_2)]$. The most prominent representatives of global dependence measures which only dependent from the underlying copula and preserve the LTD-ordering are Spearman's rank correlation coefficient $\rho(C) = 12\int_0^1\int_0^1 (C(u,v) - uv)dudv$, Kendall's rank correlation coefficient $\tau(C) = 4\int_0^1\int_0^1 C(u,v)c(u,v)dudv - 1$ and, recently introduced by Blest, $\nu(C) = 2 - 12\int_0^1\int_0^1 (1-u)^2 vc(u,v)dudv$.

Now assuming that $C(u, v)$ is a WPM copula, we face the problem of solving integrals over non-linear functions in C_1 and C_2 for $r \neq 1$. Hence, a closed and analytic form cannot be expected (as a function of r, α and C_1, C_2. Solely for $r = 1$, it is well-known that Spearman's ρ is given by $\rho(\overline{C}_1) = \alpha\rho(C_1) + (1-\alpha)\rho(C_2)$, Kendall's τ is given by $\tau(\overline{C}_1) = \alpha^2\tau(C_1) + (1-\alpha)^2\tau(C_2) + \alpha(1-\alpha)(4\int\int(C_1c_2 + C_2c_1)dudv - 1)$ and Blest's coefficient by $\nu(\overline{C}_1) = \alpha(\nu(C_1)) + (1-\alpha)\nu(C_2)$. General results for Spearman's ρ and Kendall's τ have been derived by Fischer & Hinzmann (2006) only for the weighted power mean of the independence- and the maximum copula, basically using the linear structure of these copulas.

Exemplarily, the next figure displays Spearman's rho and Kendall's tau for different WPM copulas.

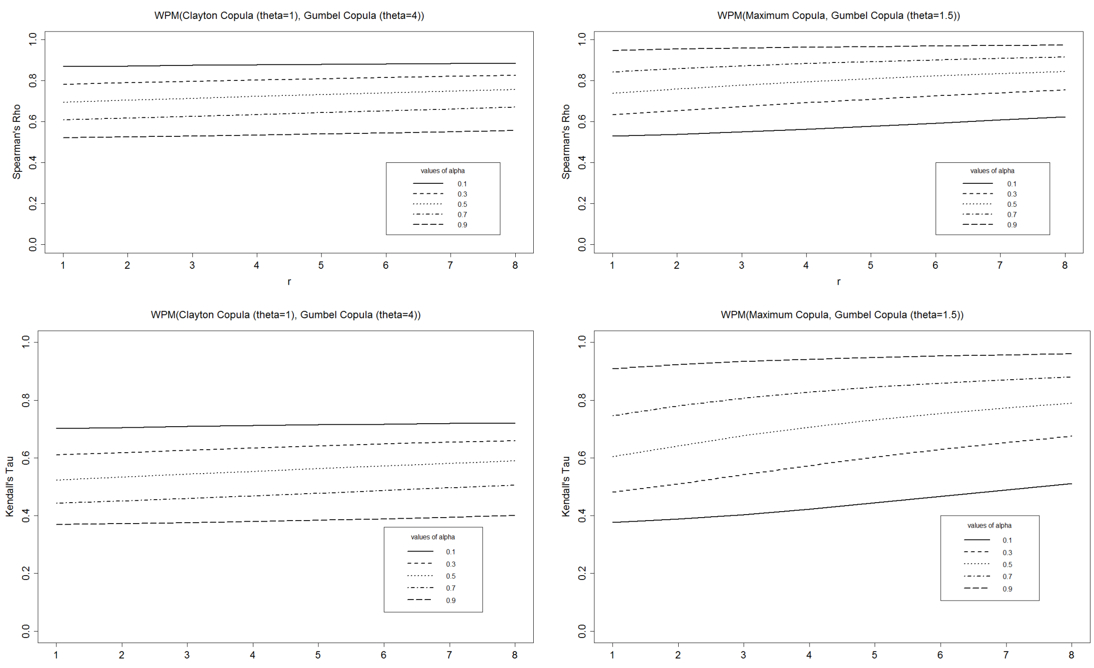


Figure 1: Spearman's rho and Kendall's tau for different WPM copulas

8 Tail dependence

8.1 Notions of tail dependence

Whereas the LTD-property deals with conditional properties of the form

$$P(U \leq u | V \leq v) \;\text{ und }\; P(V \leq v | U \leq u)$$

for varying u and v, the lower strong tail dependence coefficient (TDC) considers the asymptotic behavior of this conditional probability for $u = v$ and $u \to 0^+$:

$$\lambda_L = \lim_{u\to 0^+} P(U \le u | V \le u) = \lim_{u\to 0^+} \frac{C(u,u)}{u} = 2 - \lim_{u\to 0^+} \frac{\ln(1-2u+C(u,u))}{\ln(1-u)}.$$

Similarly, the upper strong TDC is usually defined as

$$\lambda_L = \lim_{u\to 1^-} P(U \ge u | V \ge u) = \lim_{u\to 1^-} \frac{1-2u+C(u,u)}{1-u} = 2 - \lim_{u\to 1^-} \frac{\ln C(u,u)}{\ln u}.$$

Provided their existence, these limits vary between 0 and 1. If $\lambda_U = 0$ ($\lambda_L = 0$) the pair (U, V) is commonly termed as upper (lower) strong tail independent.

In general, there are a lot of copulas (e.g. Gaussian copula, hyperbolic copula, FGM copula) which admit upper and/or lower tail independence but nevertheless allow a certain dependence between the variables U and V in the tail areas (see, e.g. Heffernan, 2000). A measure to quantify "dependence within tail independence" is suggested by Coles et al. (1999) who defines the *weak* upper tail dependence coefficient as

$$\chi_U = \lim_{u\to 1} \chi_U(u) \quad \text{with} \quad \chi_U(u) = \left(\frac{2\log(1-u)}{\log(1-2u+C(u,u))} - 1\right) \text{ for } u \in [0,1],$$

provided the existence. It can be shown that $-1 \le \chi_U \le 1$, $\chi_U = 1$ in case of upper tail dependence (i.e. for $\lambda_U > 0$), $\chi_U = 0$ in case of $C = \Pi$ being the independence copula and for copulas with upper tail independence (i.e. with $\lambda_U = 0$), χ_U increases with the strength of dependence in the tail area. In the sequel, we speak of *weak upper tail independence* if $\chi_U = 0$, and of *weak upper tail dependence* if $\chi_U \neq 0$. It should be again pointed out that it is not necessary to calculate χ_U in case of strong upper tail dependence, because then $\chi_U = 1$ holds. Instead of analyzing the limit behaviour for $u \to 1$, one usually considers the bivariate transformation $S = -1/\log U$ and $T = -1/\log V$. The variables S and T have so-called *uniform Fréchet marginal distributions* with

$$P(S > s) = P(T > s) = P(U > e^{-1/s}) = 1 - e^{-1/s} \text{ for } s > 0.$$

Applying a Taylor approximation for large s, $e^{-1/s} \approx 1 - \frac{1}{s}$ and $P(S > s) = P(T > s) \approx \frac{1}{s}$. Ledford & Tawn (1996) showed that for uniform Fréchet marginal distributions and under weak conditions

$$P(S > t, T > t) \;\approx\; \mathcal{L}(t) P(S > t)^{1/\eta} \text{ for large } t$$

holds, where $\mathcal{L}(t)$ is a slowly varying function in ∞, i.e. with $\frac{\mathcal{L}(ct)}{\mathcal{L}(t)} \to 1$ for $t \to \infty$ for each $c > 0$. Moreover, the coefficient η quantifies a weak upper tail dependence coefficient because $\chi_U = 2\eta - 1$. Furthermore, it can be shown that $\lambda_U = c$ in case of $\mathcal{L}(t) \to c$ and $\chi_U = 1$ and $\lambda_U = 0$ in case of $\chi_U < 1$. Moreover, using

$$P(S > t, T > t) = P(U > e^{-1/t}, V > e^{-1/t}) = 1 - 2e^{-1/t} + C(e^{-1/t}, e^{-1/t})$$

the relation between the uniform Fréchet marginal distributions and the copula C can be established. Thus one has to check if there is a function $\mathcal{L}(t)$ slowly varying in ∞ and a η satisfying

$$1 - 2e^{-1/t} + C(e^{-1/t}, e^{-1/t}) \;\approx\; \mathcal{L}(t) \left(\frac{1}{t}\right)^{1/\eta} \quad \text{for large } t.$$

Likewise, the weak lower tail dependence coefficient equals the limit of

$$\chi_L(u) = \frac{2\log(u)}{\log(C(u,u))} - 1$$

for $u \to 0$.

In the next step we show that the TDC of a WPM copula (provided its existence) is independent from the type of mean (i.e. r) and easily derived from the individual TDC's of C_1 and C_2.

8.2 Strong tail dependence

Theorem 6 *Assume that the strong upper (lower) tail dependence coefficients of C_1 and C_2 are given by $\lambda_{U,1}$ ($\lambda_{L,1}$) and $\lambda_{U,2}$ ($\lambda_{L,2}$), respectively. If*

$$C_r(u,v) = (\alpha C_1^r(u,v) + (1-\alpha) C_2^r(u,v))^{1/r}$$

is again a copula, the corresponding strong upper and lower TDC of $\overline{C}_r$ are given by

$$\lambda_{U,r} = \alpha\lambda_{U,1} + (1-\alpha)\lambda_{U,2} \text{ and } \lambda_{L,r} = \left(\alpha\lambda_{L,1}^r + (1-\alpha)\lambda_{L,2}^r\right)^{1/r}.$$

Proof: At first, we focus on the upper case. It holds that

$$\frac{\partial C_r(u,u;\alpha)}{\partial u} = \left(\alpha \left(\frac{C_1(u,u)}{C_r(u,u)}\right)^{r-1} \frac{\partial C_1(u,u)}{\partial u} + (1-\alpha)\left(\frac{C_2(u,u)}{C_r(u,u)}\right)^{r-1} \frac{\partial C_2(u,u)}{\partial u}\right).$$

Applying the rule of de l'Hospital, the following representation holds:

$$\begin{aligned}\lambda_{U,r} &= 2 - \lim_{u\to 1} \frac{\partial C_r(u,u;\alpha)}{\partial u} \\ &= 2 - \alpha \lim_{u\to 1}\left(\frac{C_1(u,u)}{C_r(u,u)}\right)^{r-1} \frac{\partial C_1(u,u)}{\partial u} + (1-\alpha)\lim_{u\to 1}\left(\frac{C_2(u,u)}{C_r(u,u)}\right)^{r-1} \frac{\partial C_2(u,u)}{\partial u}.\end{aligned}$$

Noting that $\lim_{u\to 1} C(u,u) = 1$ for every copula C,

$$\begin{aligned}\lambda_{U,r} &= \alpha\left(2 - \lim_{u\to 1}\frac{\partial C_1(u,u)}{\partial u}\right) + (1-\alpha)\left(2 - \lim_{u\to 1}\frac{\partial C_2(u,u)}{\partial u}\right) \\ &= \alpha\lambda_{u,1} + (1-\alpha)\lambda_{u,2}.\end{aligned}$$

Similarly, for the strong lower TDC,

$$\begin{aligned}\lambda_{L,r} &= \alpha \lim_{u\to 0}\left(\frac{C_1(u,u)}{C_r(u,u)}\right)^{r-1}\frac{\partial C_1(u,u)}{\partial u} + (1-\alpha)\lim_{u\to 0}\left(\frac{C_2(u,u)}{C_r(u,u)}\right)^{r-1}\frac{\partial C_2(u,u)}{\partial u}\\ &= \alpha \lim_{u\to 0}\left(\frac{C_1(u,u)}{C_r(u,u)}\right)^{r-1}\lambda_{L,1} + (1-\alpha)\lim_{u\to 0}\left(\frac{C_2(u,u)}{C_r(u,u)}\right)^{r-1}\lambda_{L,2}.\end{aligned}$$

Applying again de l'Hospital,

$$\lambda_{L,r} = \alpha\left(\frac{\lambda_{L,1}}{\lambda_{L,r}}\right)^{r-1}\lambda_{L,1} + (1-\alpha)\left(\frac{\lambda_{L,2}}{\lambda_{L,r}}\right)^{r-1}\lambda_{L,2}$$

and therefore

$$\begin{aligned}\lambda_{L,r} &= \lambda_{L,r}^{1-r}\left(\alpha\lambda_{L,1}^r + (1-\alpha)\lambda_{L,2}^r\right) \Longleftrightarrow (\lambda_{L,r})^r = \left(\alpha\lambda_{L,1}^r + (1-\alpha)\lambda_{L,2}^r\right) \Longleftrightarrow\\ \lambda_{L,r} &= \left(\alpha\lambda_{L,1}^r + (1-\alpha)\lambda_{L,2}^r\right)^{1/r} \square\end{aligned}$$

8.3 Weak tail dependence

In contrast to the strong TDC, it will be shown that the weak TDC depends on r (i.e. on the type of mean) to some extend. Moreover, only in case of geometric means (i.e. for $r = 0$) we also come across to the dependence of α.

Theorem 7 *Assume that C_1 and C_2 have weak lower TDC $\chi_{L,1}$ and $\chi_{L,2}$. If $\overline{C}_r(u,v)$ is again a copula, the corresponding weak lower TDC of $\overline{C}_r$ is given by*

$$\chi_L^r = \begin{cases} 2\max(\eta_1,\eta_2) - 1 & \text{for } r > 0\\ 2\frac{\eta_1\eta_2}{\alpha\eta_2+(1-\alpha)\eta_1} - 1 & \text{for } r = 0\\ 2\min(\eta_1,\eta_2) - 1 & \text{for } r < 0.\end{cases}$$

Proof: At first, assume that $r = 0$. Now,

$$\begin{aligned}C_1(1-\exp(-\frac{1}{t}), 1-\exp(-\frac{1}{t})) &= \mathcal{L}_1(t)^\alpha(1/t)^{\alpha/\eta_1},\\ C_2(1-\exp(-\frac{1}{t}), 1-\exp(-\frac{1}{t})) &= \mathcal{L}_2(t)^{(1-\alpha)}(1/t)^{(1-\alpha)/\eta_2}\end{aligned}$$

and we obtain for the weighted power mean

$$C(1-\exp(-\frac{1}{t}), 1-\exp(-\frac{1}{t})) = \mathcal{L}_1(t)^\alpha\mathcal{L}_2(t)^{(1-\alpha)}(1/t)^{(\alpha/\eta_1)+(1-\alpha)/\eta_2} = \mathcal{L}(t)\left(\frac{1}{t}\right)^{1/\eta}$$

with $\eta \equiv \frac{\eta_1\eta_2}{\alpha\eta_2+(1-\alpha)\eta_2}$. Secondly, assume that $r>0$. W.l.o.g. we assume that $\eta_1 \leq \eta_2$ and

$$C_r\left(1-e^{-1/t},1-e^{-1/t}\right) \approx \left(\alpha\mathcal{L}_1^r\left(\frac{1}{t}\right)^{r/\eta_1} + (1-\alpha)\mathcal{L}_2^r\left(\frac{1}{t}\right)^{r/\eta_2}\right)^{1/r}.$$

Hence, for large $\eta_1 \leq \eta_2$,

$$\left(\frac{1}{t}\right)^{r/\eta_1} \leq \left(\frac{1}{t}\right)^{r/\eta_2},$$

such that

$$C_r\left(1-e^{-1/t},1-e^{-1/t}\right) \approx (1-\alpha)\mathcal{L}_2\frac{1}{t}^{1/\eta_2}.$$

Consequently, $\eta=\eta_2$ and $\chi_L^r = 2\eta_2 - 1$.

Finally, assume that $r<0$ and set $s \equiv 1-e^{-1/t}$. Then

$$C_r(s,s) = \frac{C_1(s,s)C_2(s,s)}{\left(\alpha C_2^{|r|}(s,s)+(1-\alpha)C_1^{|r|}(s,s)\right)^{1/|r|}} \tag{17}$$

Now the denominator can be treated similar to the positive case (but where C_1 und C_2 are exchanged): For large t and small s,

$$C_r(s,s) \approx \frac{\mathcal{L}_1(t)\mathcal{L}_2(t)\left(\frac{1}{t}\right)^{\frac{1}{\eta_1}+\frac{1}{\eta_2}}}{\left(\alpha\mathcal{L}_2^{|r|}(t)(1/t)^{|r|/\eta_2}+(1-\alpha)\mathcal{L}_1^{|r|}(t)(1/t)^{|r|/\eta_1}\right)^{1/|r|}}. \tag{18}$$

Assume $\eta_1=\eta_2=\eta$:

$$\begin{aligned} C_r(s,s) &\approx \frac{\mathcal{L}_1(t)\mathcal{L}_2(t)\left(\frac{1}{t}\right)^{\frac{2}{\eta}}}{\left(\alpha\mathcal{L}_2^{|r|}(t)+(1-\alpha)\mathcal{L}_1^{|r|}(t)\right)^{1/|r|}\left(\frac{1}{t}\right)^{1\eta}} \\ &= \left(\frac{\alpha}{\mathcal{L}_1^{|r|}(t)}+\frac{1-\alpha}{\mathcal{L}_2^{|r|}(t)}\right)^{-1/|r|}\left(\frac{1}{t}\right)^{1/\eta} = \left(\alpha\mathcal{L}_1^r(t)+(1-\alpha)\mathcal{L}_2^r(t)\right)^{1/r}\left(\frac{1}{t}\right)^{1/\eta}. \end{aligned}$$

For $\eta_1<\eta_2$ we obtain

$$C_r(s,s) \approx \frac{\mathcal{L}_1(t)\mathcal{L}_2(t)\left(\frac{1}{t}\right)^{\frac{1}{\eta_1}+\frac{1}{\eta_2}}}{\alpha^{1/|r|}\mathcal{L}_2(t)\left(\frac{1}{t}\right)^{1/\eta_2}} = \frac{\mathcal{L}_1(t)}{\alpha^{1/|r|}}\left(\frac{1}{t}\right)^{1/\eta_1} = \alpha^{1/r}\mathcal{L}_1(t)\left(\frac{1}{t}\right)^{1/\eta_1}.$$

Finally, for $\eta_1<\eta_2$ we have

$$C_r(s,s) \approx \frac{\mathcal{L}_2(t)}{(1-\alpha)^{1/|r|}}\left(\frac{1}{t}\right)^{1/\eta_2} = (1-\alpha)^{1/r}\mathcal{L}_2(t)\left(\frac{1}{t}\right)^{1/\eta_2} \square$$

Theorem 8 *Assume that C_1 and C_2 have weak lower TDC $\chi_{L,1}$ and $\chi_{L,2}$. If $\overline{C}_r(u,v)$ is again a copula, the corresponding weak upper TDC of $\overline{C}_r$ is given by*

$$\overline{\chi}_U = 2\max\{\eta_1,\eta_2\} - 1 \ \ \textit{für } r \in \mathbb{R}$$

Proof: First note that we have for the survival copula $\hat{C}_r$ of C_r

$$\hat{C}_r\left(1-e^{-1/t}, 1-e^{-1/t}\right) = 1 - 2e^{-1/t} + C_r\left(e^{-1/t}, e^{-1/t}\right).$$

Assume at first that $r \neq 0$. In this case, $x^r \approx 1 + r(x-1)$ for $x \approx 1$ and therefore

$$C_r\left(e^{-1/t}, e^{-1/t}\right) \approx \alpha C_1\left(e^{-1/t}, e^{-1/t}\right) + (1-\alpha)C_2\left(e^{-1/t}, e^{-1/t}\right)$$

for large t. Consequently,

$$\begin{aligned}\hat{C}_r\left(e^{-1/t}, e^{-1/t}\right) &\approx \alpha\left(1-2e^{-1/t} + C_1\left(e^{-1/t}, e^{-1/t}\right)\right) + (1-\alpha)\left(1-2e^{-1/t} + C_2\left(e^{-1/t}, e^{-1/t}\right)\right)\\ &\approx \alpha\mathcal{L}_1(t)\left(\frac{1}{t}\right)^{1/\eta_1} + (1-\alpha)\mathcal{L}_2(t)\left(\frac{1}{t}\right)^{1/\eta_2}.\end{aligned}$$

1. Assume $\eta_1 = \eta_2 = \eta$. Then

$$\hat{C}_r\left(e^{-1/t}, e^{-1/t}\right) \approx \mathcal{L}(t)\left(\frac{1}{t}\right)^{1/\eta}$$

   with

$$\mathcal{L}(t) = \alpha\mathcal{L}_1(t) + (1-\alpha)\mathcal{L}_2(t).$$

2. Assume $\eta_1 < \eta_2$. Because of $1/\eta_1 > 1/\eta_2$ for large t with $1/t < 1$ we obtain

$$\left(\frac{1}{t}\right)^{1/\eta_1} < \left(\frac{1}{t}\right)^{1/\eta_2},$$

   such that

$$\hat{C}_r\left(e^{-1/t}, e^{-1/t}\right) \approx \mathcal{L}(t)\left(\frac{1}{t}\right)^{1/\eta}$$

   with $\mathcal{L}(t) = (1-\alpha)\mathcal{L}_2(t)$ and $\eta = \eta_2 = \max\{\eta_1,\eta_2\}$.

3. Assume $\eta_2 > \eta_1$. Then, obviously $\mathcal{L}(t) = \alpha\mathcal{L}_1(t)$ and $\eta = \eta_1 = \max\{\eta_1,\eta_2\}$.

Secondly, assume that $r = 0$. Using the approximations $\ln x \approx x - 1$ for $x \approx 1$ and $e^x \approx 1 + x$ for $x \approx 0$,

$$\begin{aligned}\hat{C}_r\left(e^{-1/t}, e^{-1/t}\right) &\approx 1 - 2e^{-1/t} + \exp\left(\alpha\ln C_1\left(e^{-1/t}, e^{-1/t}\right) + (1-\alpha)\ln C_2\left(e^{-1/t}, e^{-1/t}\right)\right)\\ &\approx 1 - 2e^{-1/t} + \alpha C_1\left(e^{-1/t}, e^{-1/t}\right) + (1-\alpha)C_2(\left(e^{-1/t}, e^{-1/t}\right)\\ &\approx \alpha\mathcal{L}_1(t)\left(\frac{1}{t}\right)^{1/\eta_1} + (1-\alpha)\mathcal{L}_2(t)\left(\frac{1}{t}\right)^{1/\eta_2}.\end{aligned}$$

To sum up, for all cases $\eta_1 = \eta_2$, $\eta_1 < \eta_2$ and $\eta_1 > \eta_2$ we obtain the same result for $r \neq 0$. $\square$

Example 10 *Following Fischer & Klein (2007), the weak TDC* χ_L *of the geometric mean* C *of two (arbitrary) extreme value copulas* C_1 *and* C_2 *is given by*

$$\chi_L = \frac{2 - \alpha V_1(1,1) - (1-\alpha)V_2(1,1)}{\alpha V_1(1,1) - (1-\alpha)V_2(1,1)},$$

where V_i *denotes the dependence function of* C_i *which underlies the extreme value copula* C_i*, i.e.* $C_i(u,v) = \exp\left(-V_i(-1/\ln(u), -1/\ln(u))\right)$. *Additionally,* $\chi_L = \frac{2-V_i(1,1)}{V_i(1,1)} = 2\eta_i - 1$*, i.e.* $V_i(1,1) = \frac{1}{\eta_i}$. *Hence, the dependence function* V *of* C *is given by*

$$V(1,1) = \alpha V_1(1,1) + (1-\alpha)V_2(1,1) = \alpha\frac{1}{\eta_1} + (1-\alpha)\frac{1}{\eta_2} = \frac{\alpha\eta_2 + (1-\alpha)\eta_1}{\eta_1\eta_2}.$$

and we obtain - according to the formula from Fischer & Klein (2007) -

$$\chi_L = \frac{2}{V(1,1)} - 1 = \frac{2}{\frac{\alpha\eta_2+(1-\alpha)\eta_1}{\eta_1\eta_2}} - 1 = \frac{2\eta_1\eta_2}{\alpha\eta_2 + (1-\alpha)\eta_1} - 1.$$

which corresponds to the formula from theorem 7, above.

9 Application to exchange rate data

1. *Data*: In order to illustrate the benefits of our new construction method, consider two data series from foreign exchange markets (FX-markets) which are available from the PACIFIC Exchange Rate Service (`http://pacific.commerce.ubc.ca`). In contrast to the volume notation, where values are expressed in units of the target currency per unit of the base currency, the so-called price notation is used within this study which corresponds to the numerical inverse of the volume notation. All values are expressed in units of the base currency per unit of the target currency. For reasons of brevity, we focus on the Swiss Franc/US-Dollar (SFR) and the British Pound/US-Dollar (GBP) henceforth, with daily observations from Feb 2, 1973 to Dec 31, 2009 (see figure 2). Instead of using exchange rates itself we consider log-returns (i.e. differences of log-prices), instead which are displayed in figure 3.

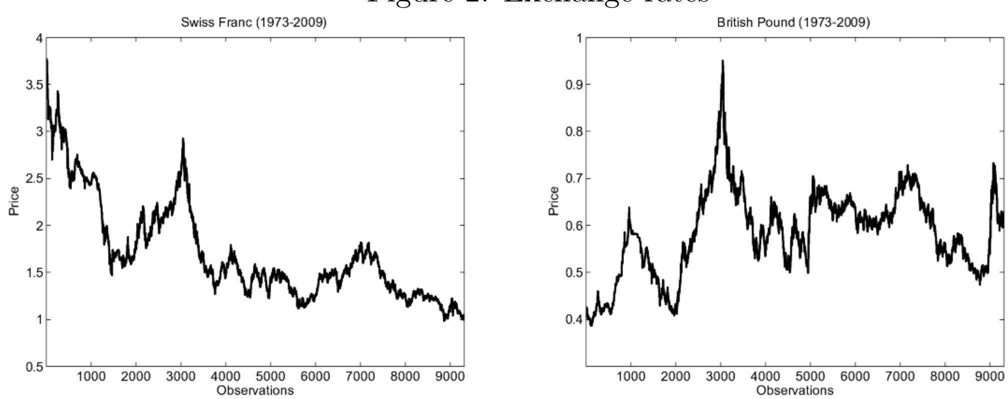
A first insight into the structure of the return series may be gained through table 2, below. It contains some basic descriptive and inductive statistics of the daily time series: the number of observations (N), arithmetic means ($Mean$), standard deviations (Std), third and fourth standardized moments ($\mathcal{S}, \mathcal{K}$). Moreover, tests on normality (i.e. Jarque-Bera test, $\mathcal{JB}$) and tests of GARCH effects (i.e. Ljung-Box test for the squared returns $\mathcal{Q}^2_{LB}$ with lag 5 and Engle's LM test $\mathcal{LM} = N \cdot \mathcal{R}^2 \overset{a}{\sim} \chi^2(k)$, where $\mathcal{R}^2$ is the coefficient of determination of the regression $R_t^2 = \alpha_0 + \alpha_1 R_{t-1}^2 + \ldots + \alpha_k R_{t-k}^2 + \varepsilon_t$), complete the summary statistics (p-values´are cited for all statistical tests).

	N	$Mean$	Std	$\mathcal{S}$	$\mathcal{K}$	$\mathcal{JB}$	$\mathcal{Q}^2_{LB}(5)$	$\mathcal{LM}(5)$
SFR	9307	−0.014	0.712	−0.055	6.257	0.000	0.000	0.000
GBP	9307	0.004	0.617	0.144	7.332	0.000	0.000	0.000

Table 2: Descriptive statistics.

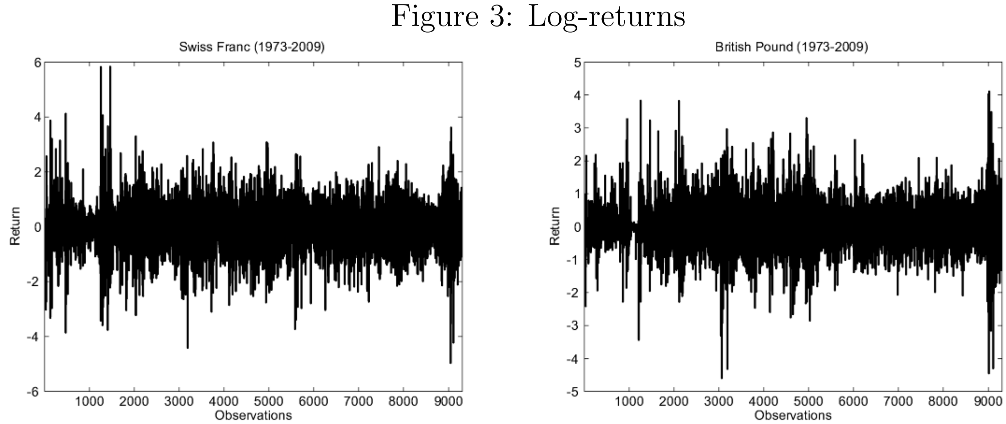
Caused by the strong evidence of GARCH effects, we apply a data filter, which means that we fitted GARCH models to the time series in a first and calculate GARCH residuals in a second step.

Figure 2: Exchange rates



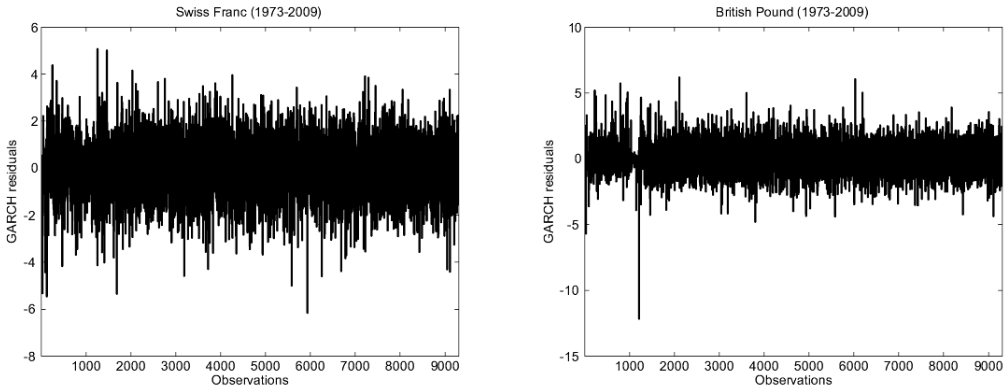
(a) SFR / USD (b) GBP / USD

Figure 3: Log-returns



(a) SFR / USD (b) GBP / USD

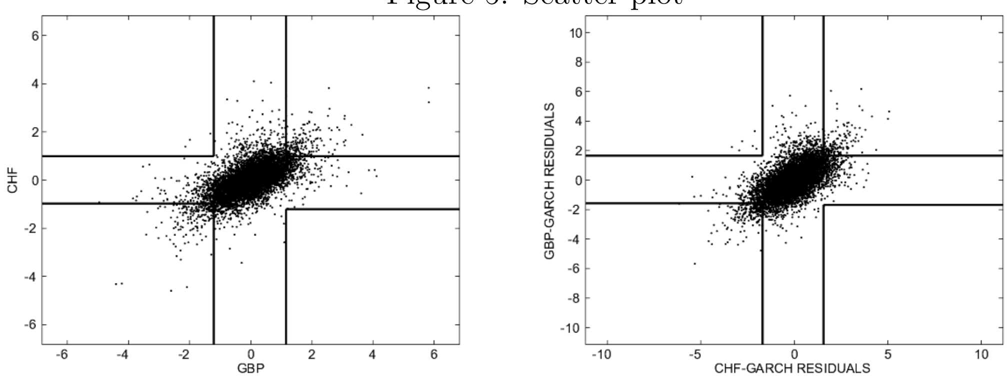
Figure 4: GARCH residuals



(a) SFR / USD (b) GBP / USD

The scatter plot for both returns and GARCH residuals is dedicated to figure 5. As to be expected, empirical correlations are essentially equal around 0.62 in both cases.

Figure 5: Scatter plot



(a) Returns ($\widehat{\rho} = 0.6249$) (b) Residuals ($\widehat{\rho} = 0.6200$)

2. *Estimation*: Within this work we apply semi-parametric maximum likelihood (SML) estimation which is treated, for example, by Hu (2002) and Genest, Ghoudi & Rivest (1995). Without any parametric assumptions for the margins, the univariate empirical cumulative distribution functions are plugged in the parametric log likelihood function, instead. In other words, after transforming the observed data pairs $(x_t; y_t)$ to uniform data pairs $(u_t; v_t)$, the SML estimator of the copula parameter θ_c maximizes the log likelihood function:

$$\theta_c = \arg\max_{\theta_c} \sum_{t=1}^{T} \log c(u_t; v_t; \theta_c).$$

3. *Copulas*: The copulas under consideration are chosen on the basis of theorem 1. Exemplarily, the Clayton copula (denoted by C_1), its rotated counterpart (C_2)

and the Gumbel copula (C_3) are selected. For reasons of brevity we introduce some abbreviations, in particular WPM($C_1, C_2; r, \alpha$) denotes the weighted power mean of the two copulas C_1 and C_2 with parameter r and α (see also table 3, below).

Abbr.	Copula	$C(u,v)$	Parameter
C1	Clayton	$(u^\theta + v^\theta - 1)^{1/\theta}$	$\theta > 0$
C2	Rotated Clayton	$u + v - 1 + ((1-u)^\theta + (1-v)^\theta - 1)^{1/\theta}$	$\theta > 0$
C3	Gumbel	$\exp(-((-\ln(u))^\theta + (-\ln(v))^\theta)^{1/\theta})$	$\theta > 1$
C4	WPM(C1,C2;r, α)	see (1)	$r > 1$
C5	WPM(C1,C3;r, α)	see (1)	$r > 1$

Table 3: Copulas under consideration

Finally, table 4 contains the estimation results from the SML estimation for both original returns and GARCH residuals. In addition to the classical log-likelihood-values ($\mathcal{LL}$) and the parameter estimators we also calculated Akaike's criterion $AIC = 2 \cdot LL + \frac{2N(k+1)}{N-k-2}$ which takes the number k of parameter into account. In both cases, the goodness-of-fit can be clearly improve for both WPM copulas under consideration. In addition, parameter estimators for GARCH residuals and return series are very similar.

Copula	$\mathcal{LL}$	AIC	$\widehat{r}$	$\widehat{\alpha}$	$\widehat{\theta}$
		Returns			
C1	2122.4	-4240.8	–	–	1.1808
C2	1896.8	-3789.6	–	–	1.1011
C3	2422.0	-4840.0	–	–	1.7634
WPM(C1,C2;r, α)	2642.4	-5274.8	1.2759	0.5484	$(1.3577, 1.8922)$
WPM(C1,C3;r, α)	2715.9	-5421.8	5.4267	0.4244	$(0.9502, 2.1446)$
		GARCH residuals			
C1	2061.1	-4118.2	–	–	1.1524
C2	1816.9	-3629.8	–	–	1.0597
C3	2319.5	-4635.0	–	–	1.7358
WPM(C1,C2;r, α)	2556.0	-5102.0	1.1780	0.5665	$(1.3621, 1.8766)$
WPM(C1,C3;r, α)	2610.5	-5211.0	3.5132	0.4164	$(1.0628, 2.0518)$

Table 4: Estimation results

10 Conclusion

In general, the weighted power mean of two arbitrary copulas is not necessary a copula again. We establish sufficient conditions which guarantee that this is true and give several examples of new copulas. Moreover, dependence measures for so-called weighted power mean (WPM) copulas are derived and calculated exemplarily for some copulas. Finally, application of WPM copulas to financial return data is given which highlight the flexibility of our new approach.

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