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Institut für Wirtschaftspolitik und Quantitative Wirtschaftsforschung

Diskussionspapier
Discussion Papers

No. 03/2012

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ISSN 1867-6707

Friedrich-Alexander-Universität
IWQW
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summary

We introduce a new skewed and leptokurtic distribution derived from the hyperbolic secant distribution and Johnson's S transformation. Properties of this new distribution are given. Finally, we empirically demonstrate in the context of financial return data that its flexibility is comparable to that of their most advanced peers.

Keywords and phrases: hyperbolic secant distribution; S_U-transformation; skewness; leptokurtosis, polynomial tails

1 Introduction and motivation

There is empirical evidence that tails of financial return distributions are so heavy that moments exist only up to a certain order, see Blattberg & Gonedes [2]. Therefore, Student-t distribution and its generalizations often come to application for both practical and theoretical reasons, see e.g. Zhu & Galbraith [32] or Rosco et al. [29]. Often, there are no simple expressions in closed form for the cumulative distribution and quantile function available (which might be useful, e.g. in the context of financial risk measure). The purpose of this paper is to overcome this shortcoming by introducing the so-called S-transformed hyperbolic secant (briefly SHS) distribution which has power tails, on the one hand, but also admits simple expression for it density, cumulative distribution and quantile function, on the other hand.

2 Skew and leptokurtic distributions by means of variable transformation

Starting from a standard normal variable X (or, more generally, from an arbitrary symmetric variable), Tukey [30] postulated requirements on a transformation function $\mathcal{T}$, such that the transformed variable $\mathcal{T}(X)$ allows for skewness and heavy tails. Examples are g and h distributions (see Hoaglin [18]), g and k distributions (see Haynes et al. [17]) or the j distribution family and its generalizations (see Klein & Fischer [21] or Fischer et al. [7]). All of them are essentially special cases of so-called generalized Tukey-type distributions (GTTD) which have been introduced and discussed by Fischer [8], [9]. Though being very flexible, evaluation of characterizing functions of GTTD's like density and cumulative distribution function requires some numerical effort (concrete: solve non-linear equations).

In contrast, if Johnson's S-transformation (see figure 1)

$$\mathcal{S}(x) \equiv \mathcal{S}_{\theta,\beta}(x) = \sinh(\theta^{-1}(x+\beta)), \quad \beta \in \mathbb{R}, \theta > 0 \tag{2.1}$$

is applied to the normal distribution (see Choi & Nam [3], Hansen et al. [15] or Rieck and Nedelmann [28]), a flexible distribution family results for which all moments exist. Instead of the normal distribution we will focus on the hyperbolic secant distribution in the next section. The hyperbolic secant distribution (HSD) has its origin in Fisher [13], Dodd [4], Roa [27] and Perks [26]. It is bell-shaped like the Gaussian distribution but has slightly heavier tails. However, in contrast, both probability density function, cumulative density function and quantile function admit simple and closed-form expressions, which makes it appealing from a practical and a theoretical point of view (see also see Fischer [9] and [12]). More precise, a random variable $X = \ln(N_1/N_2)$, where N_1, N_2 are independent standard normal variables, is said to follow a hyperbolic secant or inverse hyperbolic cosine distribution. Applying standard techniques of variable transformation, the hyperbolic secant density derives as

$$f_X(x) = \frac{1}{\pi \cosh(x)} = \frac{2}{\pi(e^{-x} + e^x)}, \quad x \in \mathbb{R}. \tag{2.2}$$

Obviously, the density is symmetrical around zero, i.e. $f(-x) = f(x)$ and has mode at zero with $f_X(0) = 1/\pi$. The corresponding cumulative distribution function of X is

$$F_X(x) = \frac{2}{\pi} \arctan(e^x). \text{ Consequently, } F_X^{-1}(p) = \ln\left(\tan\left(\frac{\pi}{2}p\right)\right). \tag{2.3}$$

All moments exist and it can be shown that the moment-generating function reads as

$$\mathcal{M}_X(t) = \mathbb{E}(e^{tX}) = \frac{1}{\cos(\pi t/2)} \quad \text{for} \quad |t| < 1.$$

In particular, the kurtosis coefficient m_4 of a hyperbolic secant variable is 5 which means that its tails are heavier than those of a normal ($m_4 = 3$) or even a logistic distribution ($m_4 = 4.2$). Notice that there are already several generalizations that allow for skewness and flexible kurtosis, all of them, however, only allow for semi-heavy tails: Examples are NEF-GHS or Meixner distribution (see Morris [25]), BHS distribution (see Fischer & Vaughan [10], SGSH_1 and SGSH_2 distribution (see Fischer [5] and [6]).

3 SHS distribution and its properties

First recall (see Mood et al. [24]), that for an arbitrary monotone transformation $\mathrm{T} : \mathbb{R} \to \mathbb{R}$, the cumulative distribution function of $X = \mathrm{T}(Z)$ is given by

$$F_X(x) = P(X \leq x) = P(\mathrm{T}^{-1}(X) \leq \mathrm{T}^{-1}(x)) = P(Z \leq \mathrm{T}^{-1}(x)) = F_Z(\mathrm{T}^{-1}(x)).$$

From that, the corresponding density reads as

$$f_X(x) = f_Z(\mathrm{T}^{-1}(x)) \left| \frac{d\mathrm{T}^{-1}(x)}{dx} \right| \tag{3.1}$$

and the quantile function as $F_X^{-1}(x) = \mathrm{T}(F_Z^{-1}(x))$.

Before we introduce the new distribution family let us point out, that

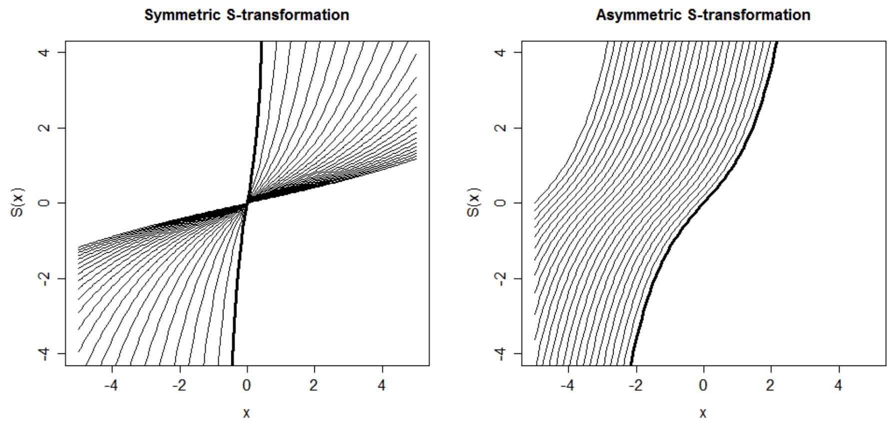

Figure 1: SHS transformation: $\theta \in [0.2, 5], \beta = 0$ (left panel) and $\theta = 1, \beta \in [0.2, 5]$ (right panel).

- the $\mathcal{S}$-transform is s.m. increasing because $\mathrm{S}'_{\theta,\beta}(x) = \theta^{-1}\cosh(\theta^{-1}(x+\beta)) > 0$.
- the inverse $\mathcal{S}$-transform is $\mathcal{S}^{-1}(x) = \theta\text{asinh}(x) - \beta$ and

$$\text{asinh}(x) = \sinh^{-1}(x) = \ln(x + \sqrt{x^2+1}) \text{ with } \text{asinh}'(x) = \frac{1}{\sqrt{x^2+1}}.$$

1. Definition and characterizing functions: Assuming then that $T = \mathcal{S}$ from (2.1), and that X is hyperbolic secant with characterizing function (2.2) and (2.3), respectively, the SHS density results:

$$f(x, \beta, \theta) = \frac{\theta}{\pi\,\cosh\left(\theta\,\text{asinh}(x) - \beta\right)\sqrt{x^2+1}}, \quad x \in \mathbb{R}. \tag{3.2}$$

Using the relationship

$$\cosh\left(\theta\,\text{asinh}(x) - \beta\right) = \frac{1}{2}\left\{\left(x + \sqrt{x^2+1}\right)^{\theta} e^{-\beta} + \left(x + \sqrt{x^2+1}\right)^{-\theta} e^{\beta}\right\}$$

we can re-write (3.2) and obtain the simple form

$$f(x, \beta, \theta) = \frac{2\theta/\pi}{\left(\left(x+\sqrt{x^2+1}\right)^{\theta} e^{-\beta} + \left(x+\sqrt{x^2+1}\right)^{-\theta} e^{\beta}\right)\sqrt{x^2+1}}. \tag{3.3}$$

Figure 2 illustrates the effect of varying β and θ.

Figure 2 to be inserted here

Its corresponding cumulative distribution function reads as

$$F(x,\beta,\theta) = \frac{2}{\pi}\arctan\left[\exp\left(\theta \operatorname{asinh}(x) - \beta\right)\right] \tag{3.4}$$

with inverse (quantile) function

$$F^{-1}(u,\beta,\theta) = \sinh\left(\frac{\ln\left(\tan(\pi u/2) + \beta\right)}{\theta}\right). \tag{3.5}$$

As it will be shown later β and θ governs both skewness and peakedness/kurtosis.

2. Tail behaviour, moments and ψ-function: For large (positive) x, the density (3.3) can be approximately re-written

$$f(x,\beta,\theta) \approx \frac{2\theta/\pi}{\left((2x)^{\theta}e^{-\beta} + (2x)^{-\theta}e^{\beta}\right)x} \approx \frac{C(\theta,\beta)}{x^{\theta+1}}.$$

Hence, SHS tails are polynomial (like Student-t tails) and moments $\mathbb{E}(X^k)$ only exist up order k which depends on the parameter θ. Details are proven in the following lemma.

Lemma 3.1 (Moments)**.** *The moments of a SHS distribution only exist up to order $k \leq \theta$. In particular,*

$$\mathbb{E}(X^n) = \frac{1}{2^n}\sum_{i=0}^{n}\binom{n}{i}\left(e^{\frac{n-2i}{\theta}\beta}\right)(-1)^i \mathcal{M}_Z\left(\frac{n-2i}{\theta}\right),$$

where

$$\mathcal{M}_Z(t) = \frac{1}{\cos(\pi t/2)} \quad \textit{for} \quad |t| < 1$$

denotes the moment-generating function of a hyperbolic secant variable.

Proof: Provided its existence, the moments of the SHS family derive as follows: For $n \in \mathbb{N}$ notice that

$$\begin{aligned}
S(z)^n &= \frac{1}{2^n}\left(e^{\theta^{-1}(z+\beta)} - e^{-\theta^{-1}(z+\beta)}\right)^n \\
&= \frac{1}{2^n}\sum_{i=0}^{n}\binom{n}{i}\left(e^{\theta^{-1}(z+\beta)(n-i)}\right)\left(e^{-\theta^{-1}(z+\beta)i}\right)(-1)^i \\
&= \frac{1}{2^n}\sum_{i=0}^{n}\binom{n}{i}\left(e^{\theta^{-1}(z+\beta)(n-2i)}\right)(-1)^i \\
&= \frac{1}{2^n}\sum_{i=0}^{n}\binom{n}{i}\left(e^{\frac{n-2i}{\theta}(z+\beta)}\right)(-1)^i = \frac{1}{2^n}\sum_{i=0}^{n}\binom{n}{i}\left(e^{\frac{n-2i}{\theta}\beta}\right)\left(e^{\frac{n-2i}{\theta}z}\right)(-1)^i.
\end{aligned}$$

Replacing z by Z and taking expectations, we obtain

$$\mathbb{E}(X^n) = \mathbb{E}(S(Z)^n) = \frac{1}{2^n}\sum_{i=0}^{n}\binom{n}{i}\left(e^{\frac{n-2i}{\theta}\beta}\right)(-1)^i\mathcal{M}_Z\left(\frac{n-2i}{\theta}\right) \square$$

Corollary 3.1. The first four power moments are given by

$$\mathbb{E}(X) = \frac{\sinh(\beta/\theta)}{\cos(0.5\pi/\theta)}, \quad \theta > 1$$

$$\mathbb{E}(X^2) = \frac{1}{2}\left(\frac{\cosh(2\beta/\theta)}{\cos(\pi/\theta)} - 1\right), \quad \theta > 2$$

$$\mathbb{E}(X^3) = \frac{1}{4}\left(\frac{\sinh(3\beta/\theta)}{\cos(1.5\pi/\theta)} - 3\frac{\sinh(\beta/\theta)}{\cos(0.5\pi/\theta)}\right), \quad \theta > 3$$

$$\mathbb{E}(X^4) = \frac{1}{8}\left(\frac{\cosh(4\beta/\theta)}{\cos(2\pi/\theta)} - 4\frac{\cosh(2\beta/\theta)}{\cos(\pi/\theta)} + 3\right), \quad \theta > 4.$$

From this, variance, skewness and kurtosis (measured by third and fourth standardized moments) can be calculated in a straightforward manner. For instance, the variance reads as

$$Var(X) = \frac{\left(\cosh\left(\frac{2b}{t}\right)\left(\cos\left(\frac{\pi}{2t}\right)\right)^2 - \cos\left(\frac{\pi}{t}\right)\left(\cos\left(\frac{\pi}{2t}\right)\right)^2 - 2\cos\left(\frac{\pi}{t}\right)\left(\cosh\left(\frac{b}{t}\right)\right)^2 + 2\cos\left(\frac{\pi}{t}\right)\right)}{2\cos\left(\frac{\pi}{t}\right)\left(\cos\left(\frac{\pi}{2t}\right)\right)^2}.$$

The following tables 2 and 1 illustrate the range of skewness and kurtosis for different parameter constellations:

Table 1 and 2 to be inserted here

Recall that ψ-functions form the basic element in the context of robust statistics, in particular of robust regression, which is an alternative to least squares regression when data are contaminated with outliers or influential observations. Concrete, by means of its finite limit, the weight of large observation is reduced. The following result can be deduced.

Lemma 3.2 (ψ-function)**.** *The ψ-function of a SHS variable (see figure 3) is given by*

$$\psi(x;\beta,\theta) = \frac{x\cosh\left(-\theta\, asinh\,(x) + \beta\right) - \theta\sinh\left(-\theta\, asinh\,(x) + \beta\right)\sqrt{x^2+1}}{(x^2+1)\cosh\left(-\theta\, asinh\,(x) + \beta\right)}, \quad x \in \mathbb{R}$$

Figure 3 to be inserted here

3. Unimodality: Finally, the unimodality of SHS distributions will be established.

Lemma 3.3 (Unimodality)**.** *All SHS densities are unimodal.*

Proof: Notice that

$$f'(x) = -\frac{\left(x\cosh\left(\theta\operatorname{asinh}(x)\right) + \theta\,\sinh\left(\theta\operatorname{asinh}(x)\right)\sqrt{x^2+1}\right)\theta}{(x^2+1)^{3/2}\left(\cosh\left(\theta\operatorname{asinh}(x)\right)\right)^2\pi}.$$

Hence, we can focus only on the denominator which reads as

$$-\left(x\cosh\left(\theta\operatorname{asinh}(x) - \beta\right) + \theta\,\sinh\left(\theta\operatorname{asinh}(x) - \beta\right)\sqrt{x^2+1}\right)\theta$$

and has first derivative

$$-\left(\underbrace{\cosh\left(\theta\operatorname{asinh}(x)-\beta\right)}_{>0}+2\underbrace{\frac{x\sinh\left(\theta\operatorname{asinh}(x)-\beta\right)\theta}{\sqrt{x^2+1}}}_{>0}+\theta^2\underbrace{\cosh\left(\theta\operatorname{asinh}(x)-\beta\right)}_{>0}\right)\theta$$

which is always negative, because of the positive parts and for $\theta > 0$. Together with the limit behaviour of $f'(x)$ the assertion follows. □

4 Fitting a SHS distribution

Assume that the underlying data are independent and identically distributed, i.e.

$$R_t = \mu + \sigma U_t \text{ with } U_t \sim f_{SHS}(\beta, \theta), \quad t = 1, \ldots, T,$$

with location parameter $\mu \in \mathbb{R}$ and (constant) scale $\sigma > 0$. Define the vector of unknown parameters as $\Theta = (\mu, \sigma, \beta, \theta)$ and suppose that N observations are $r_1, \ldots, r_N$ are given. The corresponding log-likelihood function is defined as

$$LL(\theta) = \sum_{i=1}^{N} \ln\left(f_{SHS}(r_1, \ldots, r_N; \Theta)\right).$$

Then, the maximum likelihood estimator (MLE) of Θ, indicated by $\widehat{\ell}_{ML}$ is the solution of the following optimization problem:

$$\widehat{\ell}_{ML} = \text{argmax}_{\Theta} LL(\Theta).$$

This optimization problem is solved in the empirical part using the statistical software package R, in particular using the constrained optimization function `nlminb` (see Gay [14]).

5 Application of SHS distributions to finance

1. **Data set**: To illustrate the flexibility of the new distribution, consider data from foreign exchange markets (FX-markets) which are available from the PACIFIC Exchange Rate Service[1]. This service offered by Prof. Werner Antweiler at UBC's Sauder School of Business provides access to current and historic daily exchange rates through an on-line database retrieval and plotting system. In contrast to the volume notation, where values are expressed in units of the target currency per unit of the base currency[2], the so-called *price notation* is used within this work which corresponds to the numerical inverse of the volume notation. All values are expressed in units of the base currency per unit of the target currency. Many

[1]Download under the URL-link `http://pacific.commerce.ubc.ca`.

[2]This is commonly used in Northern America to quote exchange rates.

European countries quote exchange rates this way. Daily exchange rates for the EUR-USD are available from Jan 1 2002 to Apr 30, 2012 ($n = 2593$). Figure 4 illustrates the corresponding time series for both levels and returns.

Figure 4 to be inserted here

With reference to table 3, the log-returns `EURUSD` are slightly skewed but highly leptokurtic.

Table 3 to be inserted here

As there is evidence of GARCH effects (consider Ljung-Box and Lagrange-multiplier statistic), we also focus on the GARCH residuals of the original time series, denoted `EURUSDGARCH`, hence forth.

2. **Distributions under consideration**: The main purpose of this chapter is to compare the flexibility of the SHS (or its symmtric subclass, denoted by sSHS) with that of the Student-t distribution (T) and skew generalizations (ST, see Zhu and Galbraith [32]) where moments exist only up to a certain order.

3. **Measuring goodness-of-fit**: Similar to Mittnik et al. [23], four criteria are employed to compare the goodness-of-fit of the different candidate distributions. The first is the *log-Likelihood value* (ℓ_N) obtained from the Maximum-Likelihood estimation. The ℓ_N-value can be considered as an "overall measure of goodness-of-fit and allows us to judge which candidate is more likely to have generated the data". As distributions with different numbers of parameters k are used, this is taken into account by calculating the *Akaike criterion* given by

$$AIC = -2 \cdot \ell_N + \frac{2N(k+1)}{N-k-2}.$$

The third criterion is the *Kolmogorov-Smirnov distance* as a measure of the distance between the estimated parametric cumulative distribution function, $\hat{F}$, and the empirical sample distribution, F_{emp}. It is usually defined by

$$\mathcal{K} = 100 \cdot \sup_{x\in\mathbb{R}} |F_{emp}(x) - \hat{F}(x)|. \tag{5.1}$$

Finally, the *Anderson-Darling statistic* is calculated, which weights $|F_{emp}(x) - \hat{F}(x)|$ by the reciprocal of the standard deviation of F_{emp}, namely $\sqrt{\hat{F}(x)(1-\hat{F}(x))}$, that is

$$\mathcal{AD}_0 = \sup_{x\in\mathbb{R}} \frac{|F_{emp}(x) - \hat{F}(x)|}{\sqrt{\hat{F}(x)(1-\hat{F}(x))}}. \tag{5.2}$$

Instead of just the maximum discrepancy, the second and third largest value, which is commonly termed as $\mathcal{AD}_1$ and $\mathcal{AD}_2$, are also taken into consideration. Whereas $\mathcal{K}$ emphasizes deviations around the median of the fitted distribution, $\mathcal{AD}_0, \mathcal{AD}_1$ and $\mathcal{AD}_2$ allow discrepancies in the tails of the distribution to be appropriately weighted.

4. **Empirical results**: Table 4 summarizes the estimation results.

Table 4 to be inserted here

For the leptokurtic series `EURUSD` we observe that SHS distributions clearly outperform the corresponding Student-t counterparts if we focus on $\ell_{\mathbf{N}}$, **AIC** and $\mathcal{K}$, whereas the Student-t's Anderson Darling statistics are slight lower. Both families outperform the classical Gaussian or normal distribution. In case of the GARCH residuals, which exhibit only moderate kurtosis ($m_3 = 3.8534$), the results detect the "deficits" of the SHS family which allows only for kurtosis larger than 5. In this case, Student-t or it skew version demonstrate its superiority.

6 Conclusion

A new distribution family (so-called SHS distribution) is introduced whose properties are very similar to that of (skew) Student-t distribution. In contrast to the latter, all characterizing functions have a simple and closed form. As the empirical part illustrates, the SHS distribution should be used (as alternative to Student-t versions) if the underlying data sets is highly leptokurtic ($m_4 > 5$) and skewed.

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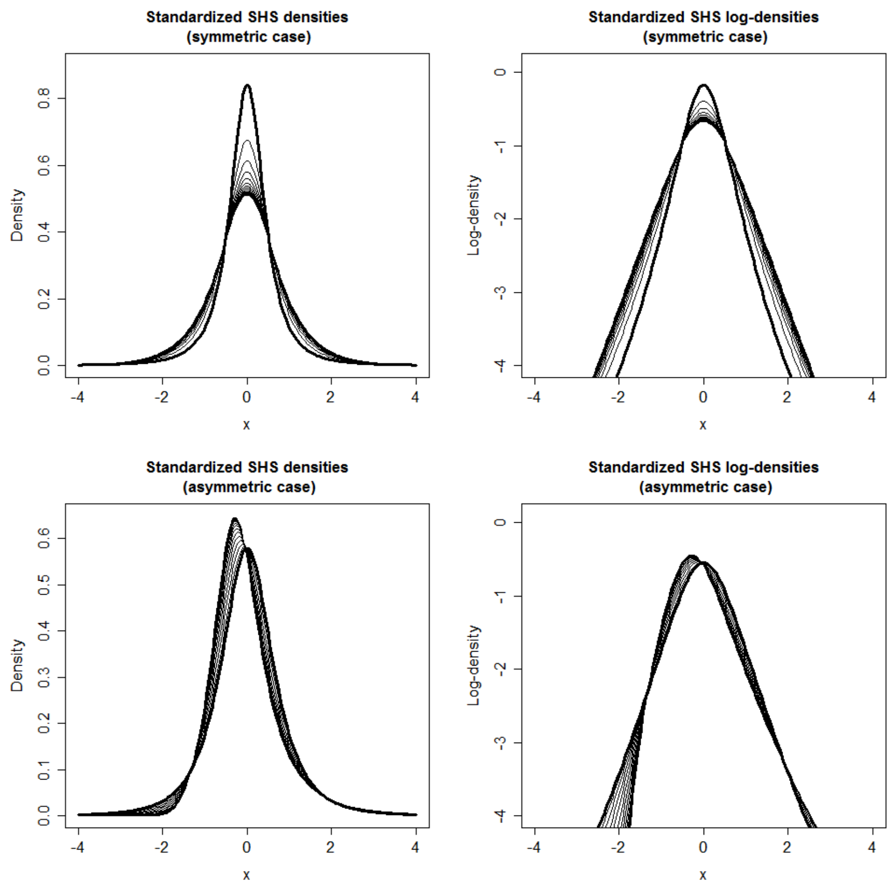

Figure 2: SHS distribution: Different densities and log-densities with $\theta \in [2.5, 10], \beta = 0$ (upper panels) and $\theta = 4$, $\beta \in [0, 10]$ (lower panels).

$\beta\downarrow,\theta\rightarrow$	4.1	4.2	4.5	5.0	5.5	6.0	8.0	8.5	9.0	9.5	10.0	15.0	20.0
0.0	78.28	41.19	18.96	11.59	9.16	7.96	6.23	6.05	5.91	5.80	5.71	5.28	5.15
0.5	84.07	43.79	19.75	11.86	9.29	8.04	6.25	6.06	5.92	5.80	5.71	5.28	5.15
1.0	100.60	51.25	22.02	12.64	9.67	8.25	6.30	6.10	5.95	5.82	5.73	5.29	5.16
1.5	125.66	62.63	25.53	13.86	10.27	8.60	6.37	6.15	5.99	5.86	5.75	5.29	5.16
2.0	156.29	76.62	29.93	15.43	11.06	9.06	6.47	6.23	6.05	5.90	5.79	5.30	5.16
2.5	189.42	91.89	34.83	17.23	11.97	9.60	6.60	6.32	6.12	5.96	5.83	5.31	5.16
3.0	222.50	107.27	39.89	19.15	12.98	10.20	6.74	6.43	6.20	6.03	5.89	5.32	5.16
3.5	253.65	121.90	44.84	21.08	14.02	10.84	6.90	6.56	6.30	6.10	5.95	5.33	5.17
4.0	281.76	135.23	49.46	22.96	15.05	11.49	7.07	6.69	6.40	6.19	6.02	5.34	5.17
4.5	306.31	146.99	53.65	24.73	16.06	12.14	7.25	6.83	6.51	6.27	6.09	5.35	5.18
5.0	327.23	157.11	57.35	26.34	17.00	12.76	7.44	6.97	6.63	6.37	6.16	5.37	5.18

Table 1: Range of kurtosis.

$\beta\downarrow,\theta\rightarrow$	4.1	4.2	4.5	5.0	5.5	6.0	8.0	8.5	9.0	9.5	10.0	15.0	20.0
0.0	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.5	0.56	0.51	0.40	0.28	0.21	0.17	0.08	0.07	0.06	0.06	0.05	0.02	0.01
1.0	1.11	1.01	0.78	0.56	0.43	0.34	0.17	0.15	0.13	0.11	0.10	0.04	0.02
1.5	1.61	1.47	1.15	0.82	0.63	0.50	0.25	0.22	0.19	0.17	0.15	0.07	0.04
2.0	2.07	1.89	1.48	1.07	0.82	0.65	0.33	0.29	0.25	0.23	0.20	0.09	0.05
2.5	2.47	2.26	1.78	1.30	1.00	0.80	0.41	0.36	0.32	0.28	0.25	0.11	0.06
3.0	2.82	2.58	2.05	1.50	1.16	0.94	0.48	0.42	0.37	0.33	0.30	0.13	0.07
3.5	3.11	2.85	2.28	1.68	1.31	1.06	0.56	0.49	0.43	0.38	0.35	0.15	0.08
4.0	3.35	3.08	2.47	1.84	1.45	1.18	0.62	0.55	0.49	0.43	0.39	0.17	0.09
4.5	3.54	3.26	2.64	1.98	1.57	1.28	0.69	0.60	0.54	0.48	0.43	0.19	0.11
5.0	3.70	3.42	2.77	2.10	1.67	1.37	0.75	0.66	0.59	0.53	0.47	0.21	0.12

Table 2: Range of skewness.

Data	No.	$\overline{X}$	S^2	$\mathbb{S}$	$\mathbb{K}$	$\mathcal{LB}$	$\mathcal{LM}$
EURUSD	2593	0.0147	0.4315	0.1012	5.4191	0.8940	0.0000
EURUSDGARCH	2591	0.0343	0.9857	-0.0116	3.8534	0.9174	0.1984

Table 3: : Descriptive and inductive data statistics.

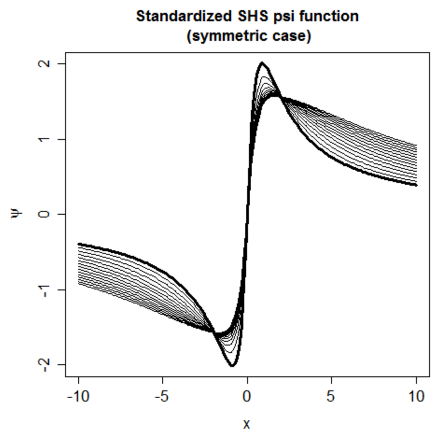

Figure 3: SHS distribution: ψ-functions for $\theta \in [3, 10], \beta = 0$.

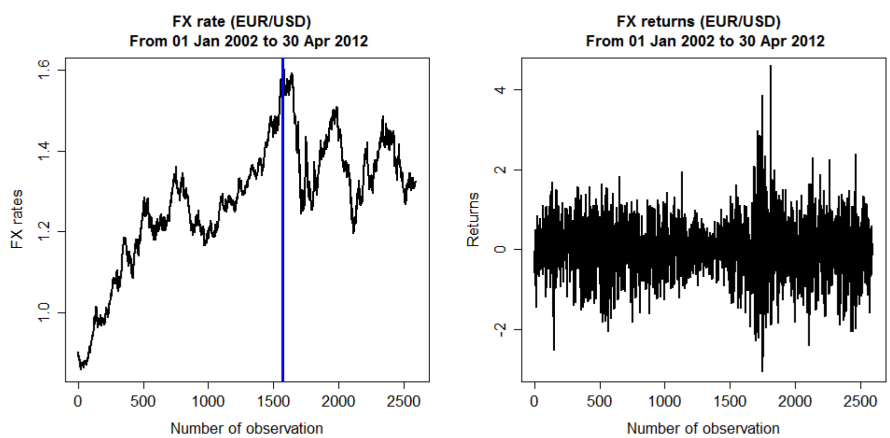

Figure 4: EUR/USD exchange rate: level versus returns

Distr.	**k**	$\ell_{\mathbf{N}}$	**AIC**	$\mathcal{K}$	$\mathcal{AD}_{\mathbf{0}}$	$\mathcal{AD}_{\mathbf{1}}$	$\mathcal{AD}_{\mathbf{2}}$
N	2	-2589.10	5184.15	4.895	7.846	0.409	0.409
T	3	-2512.22	5032.45	1.860	0.046	0.045	0.045
ST	4	-2511.70	5033.42	1.887	0.041	0.041	0.041
SHS	4	-2509.92	5029.86	1.133	0.054	0.052	0.051
sSHS	4	-2509.95	5029.92	1.141	0.056	0.054	0.053
NV	2	-3657.35	7320.71	3.360	0.083	0.074	0.072
T	3	-3640.83	7289.67	2.099	0.045	0.045	0.044
ST	4	-3640.93	7291.88	1.863	0.042	0.042	0.042
SHS	4	-3654.35	7318.73	2.005	0.080	0.079	0.079
sSHS	4	-3653.98	7317.99	2.011	0.079	0.078	0.077

Table 4: Goodness-of-fit for the unconditional and conditional case: Nikkei225.

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