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Ingo Klein
University of Erlangen-Nuremberg
Florian Christa
University of Erlangen-Nuremberg

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Families of Copulas closed under the Construction of Generalized Mean Values

Ingo Klein[1]
Florian Christa

Department of Statistics and Econometrics
University of Erlangen-Nuremberg, Germany

Abstract

We will identify sufficient and partly necessary conditions for a family of copulas to be closed under the construction of generalized linear mean values. These families of copulas generalize results well-known from the literature for the Farlie-Gumbel-Morgenstern (FGM), the Ali-Mikhai-Haq (AMH) and the Barnett-Gumbel (BG) families of copulas closed under weighted linear, harmonic and geometric mean. For these generalizations we calculate the range of Spearman's ρ depending on the choice of weights α, the copulas generating function φ and the exponent γ determining what kind of mean value will be considered. It seems that FGM and AMH generating function $\varphi(u) = 1 - u$ maximizes the range of Spearman's ρ. Furthermore, it will be shown that the considered families of copulas closed under the construction of generalized linear means have no tail dependence in the sense of Ledford & Tawn.

Keywords and phrases: copula, generalized linear means, Spearman's ρ, tail dependence

[1]Correspondence Author: Ingo Klein, Department of Statistics and Econometrics, University of Erlangen Nuremberg, D-90403 Nuremberg, Lange Gasse 20, E-Mail: ingo.klein@wiso.uni-erlangen.de

1 Stating the problem

Generalized linear means play an important role in statistics, probability and decision theory as special aggregation functions (see Grabisch et al. (2009)). Let us assume n observations $x_1, \ldots, x_n$, non-negative weights $g_1, \ldots, g_n \in \mathbb{R}$ that sum up to 1 and a strictly monotone and continuous function $u : [a, b] \to \mathbb{R}$ with inverse u^{-1}. Then

$$u^{-1}\left(\sum_{i=1}^{n} u(x_i) g_i\right), \quad g_i \geq 0, \ \sum_{i=1}^{n} g_i = 1 \tag{1}$$

is called generalized linear mean (GLM) of $x_1, \ldots, x_n$. u is said to be the generator of the GLM.

Choosing $u(x) = x^\gamma$, $\gamma \neq 0$ as generator provides the weighted power mean. For $\gamma = -1, 0, 1$ we obtain the well-known weighted harmonic, geometric and arithmetic mean.

These means can not only aggregate data but also (multivariate) distribution functions and in particular copulas, which can be viewed as distribution functions limited to the unit square $[0, 1]^2$.

If $C_1(u, v)$ and $C_2(u, v)$, $u, v \in [0, 1]$ are copulas, the weighted power mean

$$(\alpha C_1(u, v)^\gamma + (1 - \alpha) C_2(u, v)^\gamma)^{1/\gamma} \tag{2}$$

is a function on $[0, 1] \times [0, 1]$ for all $\gamma \neq 0$ and $\alpha \in (0, 1)$. Letting $\gamma \to 0$, the weighted power mean reduces to the weighted geometric mean

$$C_1(u, v)^\alpha C_2(u, v)^{1-\alpha} \tag{3}$$

of the two copulas. If (2) and (3) are copulas, they allow more flexibility for dependence modelling, due to their two additional parameters α and γ.

A systematic proof that (2) and (3) are copulas can only be found for the mean of the maximum and independence copula (see Fischer & Hinzmann (2007)). On the other hand, it is easy to construct counterexamples where the weighted mean of two copulas fails to be a copula (see f.e. Fischer et al. (2011)). For this reason, it is a non-trivial problem to identify criteria for the copulas C_1 and C_2 such that the weighted means (2) and (3) are copulas again.

This problem becomes considerably easier if we study families of copulas that are closed under the construction of means. Therefore, we know that with the two copulas C_1 and C_2 the weighted power mean of C_1 und C_2 belongs to the same family and is a copula, too.

In literature such closure properties are indeed discussed for special copula families and special means (see e.g. Nelson (1999), S. 84), but there was always a restriction to arithmetic, harmonic and geometric means. Now the question arises how copula families should be constructed so that they are closed with respect to weighted power means.

The paper is organized as follows. After a short primer on the terminology of copulas we introduce families that are closed under the construction of GLM's. Afterwards, we investigate which assumptions are necessary to assure that these potential copula functions are actually copulas and if there are additional constraints for the dependence parameter θ and the power mean parameter γ, respectively. In conclusion, some suggestions for the construction of new copula families are made based on this construction principle.

2 Copulas: An overview

For a general introduction to copulas we refer to Joe (1997), Nelsen (1999) or Drouet-Mari & Kotz (2001). In the following we just outline the most important facts on copulas that are needed throughout the paper.

We restrict ourselves to the bivariate case. If U and V have uniform distribution over $[0,1]$, we will call the restriction of the bivariate distribution function to the unit square

$$C(u,v) = P(U \leq u, V \leq v) \text{ for } u,v \in [0,1]$$

a (bivariate) copula. A Copula is mostly defined as a function $[0,1]^2 \to [0,1]$ that satisfies the boundary conditions

$$C(u,0) = C(0,v) = 0,\ C(u,1) = u,\ C(1,v) = v,\ u,v \in [0,1] \tag{4}$$

and the 2-increasing condition

$$C(u_2,v_2) - C(u_1,v_2) - C(u_2,v_1) + C(u_1,v_1) \geq 0 \tag{5}$$

for $0 \leq u_1 < u_2 \leq 1$, $0 \leq v_1 < v_2 \leq 1$.

If C is twice differentiable, one can obtain further conditions for C. The conditional probabilities

$$P(V \leq v|U \leq u) = \frac{C(u,v)}{u} \;\text{ and }\; P(U \leq u|V \leq v) = \frac{C(u,v)}{v} \tag{6}$$

and

$$\frac{\partial C(u,v)}{\partial u} = P(V \leq v|U = u) \geq 0 \;\text{ and }\; \frac{\partial C(u,v)}{\partial u} = P(V \leq v|U = u) \geq 0. \tag{7}$$

for $u, v \in [0,1]$ are dependent on C and take values in the interval $[0,1]$, which also determines the shape of C. Eventually for twice differentiable C the 2-increasing property (5) can be replaced by the condition

$$c(u,v) = \frac{\partial^2 C(u,v)}{\partial u \partial v} \geq 0 \;\text{ für }\; u, v \in [0,1], \tag{8}$$

At this, $c(u,v)$ is the so-called copula density.

3 Closure properties

Nelsen (1999) investigates whether for two copulas from the same family the mean belongs again to this family.

Considering the weighted arithmetic mean of two Farlie-Gumbel Morgenstern copulas (briefly: FGM copulas)

$$C_i(u,v) = uv + \theta_i uv(1-u)(1-v), \;\; \theta_i \in [-1,1], \; i = 1,2, \tag{9}$$

we obtain that

$$\alpha C_1(u,v) + (1-\alpha)C_2(u,v) = uv + (\alpha\theta_1 + (1-\alpha)\theta_2)uv(1-u)(1-v).$$

is again a Farlie-Gumbel-Morgenstern copula.

For two so-called Gumbel-Barnett copulas (briefly: GB copulas)

$$C_i(u,v) = uv \exp(-\theta_i \ln u \ln v), \;\; \theta_i \in (0,1], \; i = 1,2 \tag{10}$$

the weighted geometric mean

$$C_1(u,v)^\alpha C_2(u,v)^{1-\alpha} = uv \exp(-(\alpha\theta_1 + (1-\alpha)\theta_2) \ln u \ln v).$$

is again a Gumbel-Barnett copula.

The weighted harmonic mean can be used to construct copulas as well:

Regarding two Ali-Mikhail-Haq (briefly: AMH copula) copulas

$$C_i(u,v) = \frac{uv}{1-\theta_i(1-u)(1-v)}, \ \theta_i \in [-1,1], \ \ i=1,2, \tag{11}$$

the weighted harmonic mean

$$\left(\alpha\frac{1}{C_1(u,v)} + (1-\alpha)\frac{1}{C_2(u,v)}\right)^{-1} = \frac{uv}{1-(\alpha\theta_1+(1-\alpha)\theta_2)(1-u)(1-v)}$$

is again a Ali-Mikhail-Haq copula (see Nelsen (1999), p. 82).

The previous examples are special cases of a more general class of copulas which is closed under the construction of means.

Let $\varphi : [0,1] \to \mathbb{R}$ be a given function. Based on φ we regard functions C on $[0,1]\times[0,1]$ defined as

$$C(u,v;\gamma,\theta) = uv(1+\theta\varphi(u)\varphi(v))^{1/\gamma} \ \text{ for } u,v \in [0,1], \tag{12}$$

for $\gamma \neq 0$. If $\gamma = 0$, we set

$$C(u,v;\theta) = uv\exp(\theta\varphi(u)\varphi(v)) \ \text{ for } u,v \in [0,1]. \tag{13}$$

Moreover, for fixed φ we let

$$\mathcal{C}_{\varphi,\gamma} = \{C(.,.;\gamma,\theta) | \theta \in \Theta = [-1,1]\} \tag{14}$$

and

$$\mathcal{C}_{\varphi} = \{C(.,.;\theta) | \theta \in \Theta \subseteq [-1,1]\}, \tag{15}$$

be the families of potential copula functions parametrized by the depence parameter θ. [2]

Apparently, the choice of $\varphi(u) = 1-u$, $u \in [0,1]$ leads to the Farlie-Gumbel-Morgenstern family of copulas (for $\gamma = 1$) and to the Ali-Mikhail-Haq family (for $\gamma = -1$). If we let $\varphi(u) = \ln u$, $u \in [0,1]$ and $\gamma = 0$, the copula (13) leads to the Gumbel-Barnett copula with the restricted parameter space $\Theta = [-1,0)$. Cuadras (2009) also discusses the case $\varphi(u) = 1-u$, $u \in [0,1]$ and $\gamma = 0$.

[2]For (13) one can also allow unbounded functions φ if the parameter space Θ is restricted to $[-1,0)$.

If φ and γ are chosen such that (14) and (15) are parametric families of copulas, it can be shown that they are closed under the construction of generalized power means.

Propostion 1 *1. Consider $\varphi : [0,1] \to \mathbb{R}$ and $\gamma \neq 0$ with*

$$C_i(u, v; \gamma) = uv(1 + \theta_i \varphi(u)\varphi(v))^{1/\gamma} \text{ \textit{für} } u, v \in [0, 1]$$

being a copula for $\theta_i \in [-1, 1]$ and $i = 1, 2$. Then the weighted power mean of C_1 and C_2

$$(\alpha C_1(u, v; \gamma)^\gamma + (1 - \alpha)C_2(u, v; \gamma)^\gamma)^{1/\gamma} \text{ \textit{for} } u, v \in [0, 1]$$

for $\alpha \in [0, 1]$ has the same form with parameter $\alpha\theta_1 + (1 - \alpha)\theta_2$, i.e.

$$C(u, v; \gamma, \alpha\theta_1 + (1-\alpha)\theta_2) = uv(1+(\alpha\theta_1+(1-\alpha)\theta_2)\varphi(u)\varphi(v))^{1/\gamma} \text{ \textit{for} } u, v \in [0, 1].$$

2. Consider $\varphi : [0,1] \to \mathbb{R}$ with

$$C_i(u, v; \theta_i) = uv \exp(\theta_i \varphi(u)\varphi(v)) \text{ \textit{für} } u, v \in [0, 1]$$

being a copula for $\theta_i \in [-1, 1]$ and $i = 1, 2$. Then the weighted geometric mean of C_1 and C_2

$$\exp\left(\alpha \ln C_1(u, v) + (1 - \alpha) \ln C_2(u, v)\right)$$

for $\alpha \in [0, 1]$ has the same form with parameter $\alpha\theta_1 + (1 - \alpha)\theta_2$, i.e.

$$C(u, v; \alpha\theta_1 + (1 - \alpha)\theta_2) = uv \exp((\alpha\theta + (1 - \alpha)\theta_2)\varphi(u)\varphi(v)) \;\; u, v \in [0, 1].$$

Proof:

1. We have

$$\begin{aligned}&\alpha \left(uv(1 + \theta_1\varphi(u)\varphi(v))^{1/\gamma}\right)^\gamma + (1 - \alpha) \left(uv(1 + \theta_2\varphi(u)\varphi(v))^{1/\gamma}\right)^\gamma \\ &= uv^\gamma(1 + (\alpha\theta_1 + (1 - \alpha)\theta_2)\varphi(u)\varphi(v))\end{aligned}$$

   for $u, v \in [0, 1]$, $\alpha \in [0, 1]$, $\theta \in [-1, 1]$.

2. Similar to the previous proof. □

Now we have to find sufficient and preferably necessary conditions for φ and γ such that (12) and (13) are copulas for all $\theta \in [-1, 1]$ or $\theta \in [-1, 0]$. Then, due to preposition 1, the weighted power means are copulas as well.

Amblard & Girard (2002, 2003) discuss such properties for the special case $\gamma = 1$. They regard copulas of the form

$$C(u, v; \gamma = 1, \theta) = uv + \theta\phi(u)\phi(v) \quad u, v \in [0, 1] \tag{16}$$

and derive sufficient and necessary conditions for ϕ, such that (16) exhibits the properties of a copula. In this case the 2-increasing property (5) can even be proved without assuming any differentiability of ϕ. The functions φ and ϕ are related via

$$\phi(u) = u\varphi(u) \quad u \in [0, 1].$$

The conditions derived by Amblard & Girard (2002, 2003) for differentiable ϕ are mainly:

1. $\phi(u) \leq \min(u, 1 - u)$, i.e. $\varphi(u) \leq 1$ and
2. $|\phi'(u)| = |\varphi(u) + u\varphi'(u)| \leq 1$

for $u \in [0, 1]$. These conditions will re-appear in a modified manner, when we deduce conditions to assure that (12) and (13) are copulas.

Prior to that, we study how the parameter γ and θ influence the positive and negative dependence properties of the copula in the sense of Lehmann (1966). In addition, the restriction to copulas of the form (12) and (13) which admit positive or negative dependence, will also limit the choice of φ.

4 Dependence properties

A copula $C(u, v)$ is called positive (negative) dependent (see Lehmann (1966)) if

$$C(u, v) - uv \geq (\leq) 0 \quad u, v \in [0, 1].$$

For functions of the form (12) and (13) the parameter θ specifies the type of dependence according the choice of γ and φ, as long as φ doesn't change its sign on $[0, 1]$.

Propostion 2 *Let $\varphi : [0,1] \to \mathbb{R}$ be either non-negative or non-positive on $[0,1]$.*

1. *Let $C(u,v;\gamma,\theta)$ according to (12) be a copula for appropriate chosen $\gamma \neq 0$ and for all $\theta \in [-1,1]$. Then $C(u,v;\gamma,\theta)$ is positive (negative) dependent if $\gamma\theta > 0$ ($\gamma\theta < 0$).*
2. *Let $C(u,v;\theta)$ according to (13) be a copula for all $\theta \in [-1,0]$. Then $C(u,v;\theta)$ is positive (negative) dependent if $\gamma\theta > 0$ ($\gamma\theta < 0$).*

Proof:

1. Consider $\gamma > 0$. For $\theta > 0$ we have $1 + \theta\varphi(u)\varphi(v) \geq 1$ for $u, v \in [0,1]$. Hence,

   $$C(u,v;\gamma,\theta) = uv(1 + \theta\varphi(u)\varphi(v))^{1/\gamma} \geq uv \quad u,v \in [0,1],$$

   which is precisely the positive dependence property in the sense of Lehmann.

   According to this, we have negative dependence for $\theta < 0$.

   For $\gamma < 0$ the situation is reversed. Then

   $$(1 + \theta\varphi(u)\varphi(v))^{|1/\gamma|} \leq 1$$

   for $\theta < 0$, which means positive dependence due to

   $$C(u,v;\gamma) = \frac{uv}{(1 + \theta\varphi(u)\varphi(v))^{|1/\gamma|}} \geq uv \quad u,v \in [0,1],$$

   for $\theta < 0$ and vice versa.
2. Analogous. □

On the other hand, it is easy to prove that the non-positivity or non-negativity of φ is a necessary condition for the positive or negative dependence property.

Under the reasonable assumption that the copulas (12) and (13) exhibit an unique dependence property, we have to restrict ourselves to the cases where the function φ is either non-negative or non-positive on $[0,1]$.

5 Copula conditions for $\gamma \neq 0$

Firstly, we only discuss functions of the form (12).

5.1 Boundary conditions

Because every copula satisfies $C(u,1) = u$ and $C(1,v) = v$, we obtain the condition

$$\varphi(1) = 0 \tag{17}$$

for (12).

Moreover, $C(u,0) = C(0,v) = 0$ for all $u, v \in [0,1]$. Hence,

$$\lim_{u\to 0} uv(1 + \theta\varphi(u)\varphi(v))^{1/\gamma} = 0.$$

For $\gamma \geq 0$, the condition

$$\lim_{u\to 0} u^\gamma \varphi(u) = 0$$

assures that $C(0, v; \gamma, \theta) = 0$. As an example regard the function $\varphi(u) = \ln u$.

For $\gamma < 0$, the boundedness of φ is a sufficient condition, so that $C(0, v; \gamma, \theta) = 0$.

Also admitting negative γ we have to ensure that

$$1 + \theta\varphi(u)\varphi(v) \geq 0 \quad u, v \in [0,1]. \tag{18}$$

Hence,

$$(1 + \theta\varphi(u)\varphi(v))^{1/\gamma} \quad u, v \in [0,1]$$

is real-valued. In the setting $\theta = -1$ and $u = v$ this results in the condition

$$\varphi(u)^2 \leq 1 \quad \text{or} \quad |\varphi(u)| \leq 1 \quad u \in [0,1]. \tag{19}$$

For integer-valued $1/\gamma$, we can also allow unbounded functions φ such as the function $\varphi(u) = \ln u$.

Example 1 *A widely-used choice is $\varphi(u) = 1 - u$. Other specifications are e.g. $\varphi(u) = 1 - u^k,\ k > 0$.*

5.2 Conditions resulting from conditional probabilities

Note first that every differentiable copula C satisfies

$$\frac{\partial \ln C(u,v)}{\partial u} = \frac{\partial C(u,v)/\partial u}{C(u,v)} = \frac{P(V \leq v | U = u)}{C(u,v)} \geq 0 \quad u, v \in [0,1]$$

(see Nelsen (1999), p. 11). We can use the non-negativity of this partial derivative to deduce further necessary conditions.

In the case $\gamma \neq 0$ we obtain for $C(u, v; \gamma, \theta)$, $\gamma \neq 0$

$$\ln C(u, v; \gamma, \theta) = \ln u + \ln v + 1/\gamma \ln(1 + \theta\varphi(u)\varphi(v))$$

and

$$\begin{aligned} \frac{\partial \ln C(u, v; \gamma, \theta)}{\partial u} &= \frac{1}{u} + \frac{1}{\gamma} \frac{\theta\varphi'(u)\varphi(v)}{1 + \theta\varphi(u)\varphi(v)} && (20) \\ &= \frac{1 + \theta\varphi(v)(\varphi(u) + 1/\gamma u\varphi'(u))}{u(1 + \theta\varphi(u)\varphi(v))} && (21) \end{aligned}$$

for $u, v \in [0, 1]$.

With (18) and (19) the denominator is non-negative and bounded, so we have to make sure that the nominator is non-negative, too.

Due to the condition $|\varphi(v)| \leq 1$ for $v \in [0, 1]$, we have to assure that

$$|\varphi(u) + 1/\gamma u\varphi'(u)| \leq 1 \quad u \in [0, 1], \tag{22}$$

and therefore, the nominator is non-negative. Hence, $u\varphi'(u)$ has to be an bounded function on $[0, 1]$, which additionally restricts the set of admissible φ.

Example 2 *Consider*

$$\varphi(u) = \sqrt{1 - u^2} \quad u \in [0, 1].$$

This function is decreasing and concave on $[0, 1]$ *with* $\varphi(1) = 0$. *But*

$$u\varphi'(u) = -\frac{2u^2}{\sqrt{1 - u^2}}$$

is unbounded for $u \to 1$. *The same also applies for the decreasing and concave function*

$$\varphi(u) = 1 - \sqrt{1 - (1 - u)^2} \quad u \in [0, 1].$$

for $u \to 0$.

Condition (22) limits the allowed choices of γ, as the following lemma points out.

Lemma 1 *For* $u \in [0,1]$ *with* $\varphi'(u) > 0$ *we obtain*

$$|\varphi(u) + 1/\gamma u\varphi'(u)| \leq 1 \iff \frac{-1+\varphi(u)}{u\varphi'(u)} \leq \frac{1}{\gamma} \leq \frac{1-\varphi(u)}{u\varphi'(u)}. \tag{23}$$

Conversely, for $u \in [0,1]$ *with* $\varphi'(u) < 0$ *we have*

$$|\varphi(u) + 1/\gamma u\varphi'(u)| \leq 1 \iff \frac{1-\varphi(u)}{u\varphi'(u)} \leq \frac{1}{\gamma} \leq \frac{-1+\varphi(u)}{u\varphi'(u)}. \tag{24}$$

Proof: Simple rearrangement. □

For given φ, we designate with G_φ the set of all potential γ, that satisfy the bounds (23) and (24). I.e

$$G_\varphi = \{\gamma \neq 0 || \varphi(u) + 1/\gamma u\varphi'(u)| \leq 1 \;\; u \in [0,1]\}. \tag{25}$$

It is not at all guaranteed that $G_\varphi \neq \emptyset$ as can be seen in example 2.

Note that the terms

$$-\frac{u\varphi'(u)}{1-\varphi(u)} \text{ and } \frac{u\varphi'(u)}{-1+\varphi(u)}$$

can be interpreted as elasticity measures for the functions $1-\varphi$ and $-1+\varphi$ in $u \in [0,1]$. In the following example we study functions that exhibit constant elasticity and therefore allow a very simple determination of the bounds (23) and (24) for $1/\gamma$.

Example 3 *We focus on*

$$\varphi(u) = 1 - u^k, \;\; u \in [0,1], \; k > 0 \tag{26}$$

again. With $\varphi'(u) = -ku^{k-1} \leq 0$ *and*

$$\frac{1-\varphi(u)}{u\varphi'(u)} = -\frac{1}{k} \;\; u \in [0,1],$$

we have

$$-\frac{1}{k} \leq \frac{1}{\gamma} \leq \frac{1}{k}. \tag{27}$$

For $k = 1$ *we obtain the bounds* $\gamma = 1$ *and* $\gamma = -1$ *which lead to the FGM and AMH copula, respectively. Apparently, copulas of the form (12) with* $\varphi(u) = 1 - u$, $u \in [0,1]$ *can only be found within these bounds.*

Example 4 *We can also prove constant elasticity for the function*

$$\varphi(u) = \min\left(1, \frac{1}{u^k} - 1\right), \quad u \in [0,1],\ k > 0.$$

We have

$$u\varphi'(u) = \begin{cases} 0 & \text{für } u < (1/2)^{1/k} \\ -ku^{-k} & \text{für } u < (1/2)^{1/k}. \end{cases}$$

For $u < (1/2)^{1/k}$ *the condition*

$$|\varphi(u) + \frac{1}{\gamma}u\varphi'(u)| = 1 \leq 1$$

trivially holds. Thus, the domain of $1/\gamma$ *is resticted by*

$$\frac{1-\varphi(u)}{u\varphi'(u)} = -\frac{1}{k} \leq \frac{1}{\gamma} \leq \frac{-1+\varphi(u)}{u\varphi'(u)} = \frac{1}{k}$$

for $u > (1/2)^{1/k}$.

Example 5 *With*

$$\varphi(u) = e^{-u} - e^{-1} \quad u \in [0,1] \tag{28}$$

we regard another function of constant elasticity. As the lower bound of $1/\gamma$ *we obtain the function*

$$\frac{1-\varphi(u)}{u\varphi'(u)} = \frac{1-e^{-u}+e^{-1}}{-ue^{-u}} = \frac{1}{u} + \frac{1+e^{-1}}{-ue^{-u}},$$

attaining its maximum in amount of -2.487390 *in the point* $u = 0.5979717$*, as can be shown via numerical optimization.*

Furthermore,

$$\frac{-1+\varphi(u)}{u\varphi'(u)} = \frac{-1+e^{-u}-e^{-1}}{-ue^{-u}} = -\frac{1}{u} + \frac{-1-e^{-1}}{-ue^{-u}}$$

is decreasing on $[0,1]$. *We obtain that* $u = 1$ *minimizes the upper bound for* $1/\gamma$ *and hence,*

$$-2.48739 \leq \frac{1}{\gamma} \leq 2.718282 = e \tag{29}$$

must hold.

Example 6 *We can also admit non-monotone function $\varphi(u)$ on $[0,1]$ such as*

$$\varphi(u) = u(1-u) \;\; u \in [0,1].$$

The first derivative $\varphi'(u) = 1 - 2u$ is negative for $u > 1/2$ and positive for $u < 1/2$. It is easy to prove that condition (23) reads as

$$\frac{-1 + u(1-u)}{u(1-2u)} \leq -6.464 \leq \frac{1}{\gamma} \leq 6.464 \leq \frac{1 - u(1-u)}{u(1-2u)}$$

for $u < 1/2$ and that (24) reads as

$$\frac{-1 + u(1-u)}{u(1-2u)} \leq 0 \leq \frac{1}{\gamma} \leq 1 \leq \frac{1 - u(1-u)}{u(1-2u)}$$

for $u > 1/2$. Combining these expressions

$$0 \leq \frac{1}{\gamma} \leq 1$$

must hold.

5.3 2-increasing condition

5.3.1 Case: φ monotone and $\theta\gamma > 0$

Up to now we derived the necessary conditions

1. $\varphi(1) = 0$,
2. $|\varphi(u)| \leq 1$, $u \in [0,1]$ and
3. $|\varphi(u) + 1/\gamma u \varphi'(u)| \leq 1$, $u \in [0,1]$.

Now it is to be checked, which further conditions are needed to assure that the second mixed derivative $c(.,.,\gamma,\theta)$ of the potential copula is non-negative and therefore a copula density. There is a close relation between this an the second mixed derivative of the logarithmized potential copula function:

$$\frac{c(u,v;\gamma,\theta)}{C(u,v;\gamma,\theta)} = \frac{\partial^2 \ln C(u,v;\gamma,\theta)}{\partial u \partial v} + \frac{\partial \ln C(u,v;\gamma,\theta)}{\partial u} \frac{\partial \ln C(u,v;\gamma,\theta)}{\partial v}, \quad u,v \in [0,1]. \tag{30}$$

Assumption (22) makes sure that

$$\frac{\partial \ln C(u,v;\gamma,\theta)}{\partial u} \geq 0 \ \text{ and } \ \frac{\partial \ln C(u,v;\gamma,\theta)}{\partial v} \geq 0 \ \ u,v \in [0,1].$$

If additionally

$$\frac{\partial^2 \ln C(u,v;\gamma,\theta)}{\partial u \partial v} \geq 0 \ \ u,v \in [0,1] \tag{31}$$

holds, then $c(u,v;\gamma,\theta) \geq 0$ for $u,v \in [0,1]$ and $C(.,.;\gamma,\theta)$ is a copula [3]

Hence, for monotone functions φ the sufficient condition $\gamma\theta > 0$ assures that $C(.,.;\gamma,\theta)$ is a copula, as we state in the following proposition.

Propostion 3 *Let $\varphi : [0,1] \to \mathbb{R}$ be a differentiable and monotone function on $[0,1]$, fulfilling the conditions*

1. $\varphi(1) = 0$,
2. $|\varphi(u)| \leq 1$ *for* $u \in [0,1]$,
3. $G_\varphi \neq \emptyset$ *with*

$$G_\varphi = \{g \neq 0 || \varphi(u) + 1/\gamma u \varphi'(u)| \leq 1, \ u \in [0,1]\}.$$

Then

$$C(u,v;\gamma,\theta) = uv(1+\theta\varphi(u)\varphi(v))^{1/\gamma} \ \ u,v \in [0,1], \ g \in G_\varphi, \ \theta \in [-1,1]$$

is a copula if for all $g \in G_\varphi$ and $\theta \in [-1,1]$ the condition

$$\gamma\theta > 0 \tag{32}$$

holds.

Proof: After some algebra we have

$$\frac{\partial \ln C(u,v;\gamma,\theta)}{\partial u \partial v} = \frac{1}{\gamma}\frac{\theta\varphi'(u)\varphi'(v)}{(1+\theta\varphi(u)\varphi(v))^2}. \tag{33}$$

[3] With property (31), the potential copula (12) is maximum infinitely divisible (briefly: max-id), i.e. $C(.,.;\gamma))^r$ is a proper distribution function for all $r > 0$ (see Joe (1997), p. 32f.).

If φ is monotone and φ' does not change its sign on $[0,1]$, we obtain

$$\frac{\partial \ln C(u,v;\gamma,\theta)}{\partial u \partial v} \geq 0 \quad u,v \in [0,1],$$

for $\gamma\theta \geq 0$. □

To induce positive dependence for $\gamma > 0$, the parameter θ has to be non-negative. In the reversed case, θ has to be non-positive for positive dependence. Hence, in this case, the question whether (12) is a copula, can be answered very easily.

Example 7 *The function*

$$C(u,v;\gamma,\theta) = uv(1+\theta(1-u^k)(1-v^k))^{1/\gamma} \quad u,v \in [0,1]$$

is a copula for $k > 0$, if

1. $0 < \theta \leq 1$ *and* $0 < 1/\gamma \leq 1/k$ *or*
2. $-1 \leq \theta \leq 0$ *and* $-1/k \leq 1/\gamma < 0$.

In the case $k = 1$ we have the combinations $0 < \theta \leq 1$ and $0 < \gamma \leq 1$ or $-1 \leq \theta < 0$ and $-1 \leq \gamma < 0$ that include the FGM copula for $\gamma = 1$ and the AMH copula for $\gamma = -1$ as limiting cases. The same result also holds for $\varphi(u) = u^k - 1$, $u \in [0,1]$.

Example 8 *The function*

$$C(u,v;\gamma,\theta) = uv(1+\theta(e^{-u}-e^{-1})(e^{-v}-e^{-1}))^{1/\gamma} \quad u,v \in [0,1]$$

is a copula for

1. $0 < \theta \leq 1$ *and* $0 < 1/\gamma \leq 2.718282$ *or*
2. $-1 \leq \theta \leq 0$ *and* $-2.487390 \leq 1/\gamma < 0$.

5.3.2 Case: φ monotone and $\gamma\theta < 0$

Inserting expressions (21) and (33) for derivatives of the logarithmized copula we get

$$\frac{c(u,v;\gamma,\theta)}{C(u,v;\gamma,\theta)} = \frac{Z}{uv(1+\theta\varphi(u)\varphi(v))^2} \quad u,v \in [0,1]$$

with

$$Z = \frac{\theta}{\gamma} uv\varphi'(u)\varphi'(v)$$
$$+(1+\theta\varphi(v)(\varphi(u)+1/\gamma u\varphi'(u)))(1+\theta\varphi(u)(\varphi(v)+1/\gamma v\varphi'(v))) \quad (34)$$

for $u, v \in [0,1]$. Obviously, for given γ and $\theta \in [-1,1]$ the expression $c(.,.;\gamma,\theta)$ is a copula density if and only if $Z \geq 0$ for all $u, v \in [0,1]$.

We have to suppose the monotonicity of φ in order that (12) is a copula for $\theta \in [-1,1]$. Hence, $\varphi(1) = 0$ implies that the function φ is either non-negative and monotonly decreasing on $[0,1]$ or non-positive and monotonely increasing. In the case of a non-monotone φ no general result can be achieved and every function φ has to be checked separately, whether it provides a copula or not.

The following proposition states necessary conditions on that $C(.,.;\gamma,\theta)$ is a copula for an arbitrary $\gamma \neq 0$.

Propostion 4 *Let $\varphi : [0,1] \to \mathbb{R}$ be a monotone and differentiable function fulfilling the conditions*

1. $\varphi(1) = 0$,
2. $|\varphi(u)| \leq 1$ *for* $u \in [0,1]$,
3. $G_\varphi \neq \emptyset$ *with*

$$G_\varphi = \{\gamma \neq 0 | |\varphi(u) + 1/\gamma u\varphi'(u)| \leq 1,\ u \in [0,1]\},$$

4. φ *non-negative and monotonely decreasing or non-positive and monotonely increasing on* $[0,1]$.

Then

$$C(u,v;\gamma,\theta) = uv(1+\theta\varphi(u)\varphi(v))^{1/\gamma}\ \ u,v \in [0,1],\ g \in G_\varphi,\ \theta \in [-1,1]$$

is a copula, if either

$$\gamma \in G_\varphi \cap (0,1] \ \ \textit{and} \ \ \theta \in [-1,0) \quad (35)$$

or

$$\gamma \in G_\varphi \cap (-\infty,-1] \ \ \textit{and} \ \ \theta \in (0,1] \quad (36)$$

holds.

Proof: Note first, that we can restrict ourselves in the following to functions φ that are non-negative and monotonely decreasing on $[0,1]$. In the complementary case $\varphi(u) \le 0$ and $\varphi'(u) \ge 0$ we can re-write the copula function as

$$uv(1+\theta\varphi(u)\varphi(v))^{1/\gamma} = uv(1+\theta(-\varphi(u))(-\varphi(v)))^{1/\gamma}$$

where $-\varphi(u) \ge 0$ and $d(-\varphi(u))/du \le 0$.

We check the two-increasing property under the condition (35). For the second mixed derivative of the log-density we obtain

$$\begin{aligned}\frac{c(u,v,\theta,\gamma)}{C(u,v;\theta,\gamma)} &= \frac{1}{uv(1+\theta\varphi(u)\varphi(v))}\left(1+\theta\varphi(v)(\varphi(u)+1/\gamma u\varphi'(u))\right.\\ &\left.+\theta/\gamma v\varphi'(v)\left(\varphi(u)+\frac{1+\theta/\gamma\varphi(u)\varphi(v)}{1+\theta\varphi(u)\varphi(v)}u\varphi'(u)\right)\right) \qquad (37)\end{aligned}$$

where $c(u,v;\theta,\gamma)$ is the mixed second derivative of $C(u,v;\theta,\gamma)$.

With the conditions $\varphi(u) \ge 0$, $\varphi'(u) \le 0$, $|\varphi(u)+1/\gamma u\varphi'(u)| \le 1$ and $\gamma \in (0,1]$ we get

$$\frac{1+\theta/\gamma\varphi(u)\varphi(v)}{1+\theta\varphi(u)\varphi(v)} = \frac{1}{\gamma}\frac{\gamma+\theta\varphi(u)\varphi(v)}{1+\theta\varphi(u)\varphi(v)} \le \frac{1}{\gamma}.$$

Due to $\varphi'(u) \le 0$

$$\varphi(u)+1/\gamma u\varphi'(u) \le \varphi(u)+\frac{1+\theta/\gamma\varphi(u)\varphi(v)}{1+\theta\varphi(u)\varphi(v)}u\varphi'(u).$$

Multiplication with $\theta\varphi'(v) \ge 0$ leads to

$$\theta\varphi'(v)(\varphi(u)+1/\gamma u\varphi'(u)) \le \theta\varphi'(v)\left(\varphi(u)+\frac{1+\theta/\gamma\varphi(u)\varphi(v)}{1+\theta\varphi(u)\varphi(v)}u\varphi'(u)\right)$$

and therefore we obtain with (37) and $|\varphi(u)+1/\gamma u\varphi'(u)| \le 1$

$$\begin{aligned}\frac{c(u,v,\theta,\gamma)}{C(u,v;\theta,\gamma)} &\ge \frac{1}{uv(1+\theta\varphi(u)\varphi(v))}\left(1+\varphi(v)\theta\left(\varphi(u)+1/\gamma u\varphi'(u)\right)\right.\\ &\left.+1/\gamma v\varphi'(v)\theta\left(\varphi(u)+1/\gamma u\varphi'(u)\right)\right)\\ &= \frac{1}{uv(1+\theta\varphi(u)\varphi(v))}\left(1+\theta\left(\varphi(u)+1/\gamma u\varphi'(u)\right)\left(\varphi(v)+1/\gamma v\varphi'(v)\right)\right) \ge 0.\end{aligned}$$

Now prove the two-increasing property under the condition (35).

As mentioned above the second mixed derivative of the log-density can be written as

$$\frac{c(u,v;\theta)}{C(u,v;\theta)} = \frac{Z}{uv} \quad u,v \in [0,1]$$

with

$$Z=\frac{\theta}{\gamma}u\varphi'(u)v\varphi'(v)+(1+\theta\varphi(v)(\varphi(u)+1/\gamma\varphi'(u))(1+\theta\varphi(u)(\varphi(v)+1/\gamma v\varphi'(v))\quad u,v\in[0,1].$$

We suppose again $\varphi(u)\geq 0$, $\varphi'(u)\leq 0$, $|\varphi(u)+1/\gamma u\varphi'(u)|\leq 1$.

With $\gamma<0$

$$\varphi(u)+1/\gamma u\varphi'(u)\geq 0$$

and with $\theta>0$

$$(1+\theta\varphi(v)(\varphi(u)+1/\gamma u\varphi'(u)))(1+\theta\varphi(u)(\varphi(v)+1/\gamma v\varphi'(v)))\geq 1.$$

Apparently if

$$\left|\frac{u\varphi'(u)}{\sqrt{|\gamma|}}\right|\leq 1\ \text{ or }\ |u\varphi'(u)|\leq\sqrt{|\gamma|}\tag{38}$$

can be guaranteed, then $Z\geq 0$ and the two-increasing property holds.

Indeed, (38) is valid for $|\gamma|>1$ since

$$\varphi(u)\leq 1\ \text{ and }\ \varphi(u)+1/\gamma u\varphi'(u)\leq 1$$

leads to $1/|\gamma||u'\varphi(u)|\leq 1$. With $|\gamma|>1$ we obtain

$$|u\varphi'(u)|\leq\sqrt{|\gamma|}\leq|\gamma|$$

and therefore C represents a copula. □

Example 9 *We consider the function $\varphi(u)=1-u^k$, $k>0$. The Condition $|\varphi(u)+1/\gamma u\varphi'(u)|<1$ leads to*

$$\begin{aligned}\varphi(u)+1/\gamma u\varphi'(u) &= 1-(1-k/\gamma)u^k\geq -1\\ &\iff k/\gamma\geq -1\iff k\leq-\gamma=|\gamma|.\end{aligned}$$

Furthermore

$$\frac{|u\varphi'(u)|}{\sqrt{|\gamma|}}=\frac{ku^k}{\sqrt{|\gamma|}}\leq 1\iff k\leq\sqrt{|\gamma|}.$$

In summary,

$$k\leq\begin{cases}|\gamma| & \textit{for }\ -1\leq\gamma<0\\ \sqrt{|\gamma|} & \textit{for }\ \gamma<-1\end{cases}$$

must hold. Hence, for $0 \le \theta \le 1$ we obtain the copulas

$$C(u,v) = uv(1+\theta(1-\sqrt{u})(1-\sqrt{v}))^{-2}$$

with $k = 1/2$ and $\gamma = -1/2$ and

$$C(u,v) = uv(1+\theta(1-u^2)(1-v^2))^{-1/4}$$

with $k = 2$ and $\gamma = -4$.

Example 10 *The choice $\varphi(u) = \min\left(1, \frac{1}{u^k} - 1\right)$, $u \in [0,1]$, $k > 0$ leads to*

$$C(u,v;\gamma,\theta) = uv\left(1+\theta\min\left(1,\frac{1}{u^k}-1\right)\min\left(1,\frac{1}{v^k}-1\right)\right)^{1/\gamma} \quad u,v \in [0,1].$$

This is a copula for $\theta \in [-1,1]$ and for

$$-1 \le -\frac{1}{k} \le \frac{1}{\gamma} \le \frac{1}{k} \le 1,$$

i.e. for $k \ge 1$. For $\gamma = 1$ we must choose $k = 1$ and hence

$$u\varphi(u) = \min\left(1, \frac{1}{u} - 1\right) = \min(u, 1-u) \quad u \in [0,1].$$

The setting $k = 1$ represents the upper bound for $u\varphi(u)$, $u \in [0,1]$ according to Amblard & Girard (2002) which maximizes the area below the graph of $\min\left(1, \frac{1}{u} - 1\right)$, $u \in [0,1]$ for all admissible k.

Example 11 *For $\varphi(u) = e^{-u} - e^{-1}$, $u \in [0,1]$ the expression*

$$C(u,v;\gamma,\theta) = uv(1+\theta(e^{-u}-e^{-1})(e^{-v}-e^{-1}))^{1/\gamma} \quad u,v \in [0,1]$$

represents a copula for $\theta \in [-1,1]$ and

$$-2.48739 \le -1 \le \frac{1}{\gamma} \le 1 \le 2.718282.$$

6 Copula conditions for $\gamma = 0$

For $\gamma = 0$ we have defined

$$C(u,v;\theta) = uve^{\theta\varphi(u)\varphi(v)} \quad u,v \in [0,1]$$

for $\theta \in \Theta \subseteq [-1, 1]$. The concrete amount of the parameter space Θ is dependent on the properties of the function φ. We assume again that φ does not change its sign on $[0, 1]$ in order to obtain an uniquely determined dependence structure. Note, that a copula is limited to the range $[0, 1]$ and therefore, $e^{\theta\varphi(u)\varphi(v)}$ has to be bounded on $[0, 1]$.

Hence, we obtain:

1. For unbounded φ on $[0, 1]$, the function

   $$e^{\theta\varphi(u)\varphi(v)} \quad u, v \in [0, 1]$$

   is only bounded if $\theta \in [-1, 0]$. I.e. $\Theta = [-1, 0]$.

   As an example consider the function $\varphi(u) = \ln u$, $u \in [0, 1]$, which leads to the BG copula.

2. If φ is bounded on $[0, 1]$, we can allow the parameter space $\Theta = [-1, 1]$. W.l.o.g. we can assume that for bounded functions φ

   $$|\varphi(u)| \leq 1, \quad u \in [0, 1]$$

   holds. If necessary, this can be achieved via appropriate scaling.

   An example for such a bounded function φ is given by $\varphi(u) = 1 - u$ for $u \in [0, 1]$ leading to the ,,New Copula" suggested by Cuadras (2009).

In the following we have to distinguish the cases of bounded and unbounded functions φ.

Conditions resulting from conditional probabilities The first partial derivative of $\ln C(u, v; \theta)$ is

$$\frac{\partial \ln C(u, v; \theta)}{\partial u} = \frac{1}{u} + \theta\varphi'(u)\varphi(v) = \frac{1 + \theta u\varphi'(u)\varphi(v)}{u} \quad u, v \in [0, 1].$$

This is non-negative if

$$1 + \theta u\varphi'(u)\varphi(v) \geq 0 \quad u, v \in [0, 1]. \tag{39}$$

1. If φ is non-negative and monotonly decreasing or non-positive and monotonly increasing on $[0, 1]$, this condition is satisfied for $\theta < 0$.

2. For $\theta > 0$ the function φ has to be bounded, i.e. $|\varphi(v)| \leq 1$, $v \in [0,1]$. For $\theta = 1$

$$1 + u\varphi'(u) \geq 0 \iff u\varphi'(u) \geq -1 \quad u \in [0,1] \tag{40}$$

holds.

2-increasing condition The second mixed partial derivative of $\ln C(u,v;\theta)$ is given by

$$\frac{\partial^2 \ln C(u,v;\theta)}{\partial u \partial v} = \theta\varphi'(u)\varphi'(v) \quad u,v \in [0,1] \tag{41}$$

for $\theta \in \Theta$, at which we have as parameter space either $\Theta = [-1,1]$ or $\Theta = [0,1]$ depending on whether φ is bounded or not.

For $\theta > 0$ and for a monotone function φ, this second derivative is non-negative on$[0,1]$.

As a consequence we obtain the following proposition.

Propostion 5 *Let $\varphi : [0,1] \to \mathbb{R}$ be differentiable and monotone on $[0,1]$ fulfilling the conditions*

1. $\varphi(1) = 0$,
2. $|\varphi(u)| \leq 1$, $u \in [0,1]$.

Then

$$C(u,v;\theta) = uve^{\theta\varphi(u)\varphi(v)} \quad u,v \in [0,1]$$

is a copula for $\theta \in [0,1]$.

Proof: With (40) the condition $|\varphi(u)| \leq 1$ ensures that

$$\frac{\partial \ln C(u,v;\theta)}{\partial u} \geq 0 \text{ and } \frac{\partial \ln C(u,v;\theta)}{\partial v} \geq 0$$

for $u,v \in [0,1]$. Hence,

$$\frac{c(u,v;\theta)}{C(u,v;\theta)} = \frac{\partial^2 \ln C(u,v;\theta)}{\partial u \partial v} + \frac{\partial \ln C(u,v;\theta)}{\partial u}\frac{\partial \ln C(u,v;\theta)}{\partial v} \geq 0$$

for $u,v \in [0,1]$ if

$$\frac{\partial^2 \ln C(u,v;\theta)}{\partial u \partial v} \geq 0 \quad u,v \in [0,1].$$

This holds for $\theta > 0$ as (41) shows. □

Hence, for monotone functions φ the case $\theta < 0$ is of interest. For this, the second mixed partial derivative $c(.,.;\theta)$ of $C(.,.;\theta)$ is required. We obtain

$$\frac{c(u,v;\theta)}{C(u,v;\theta)} = \frac{Z}{uv} \quad u,v \in [0,1]$$

with

$$Z = \theta u\varphi'(u)v\varphi'(v) + (1+\theta u\varphi'(u)\varphi(v))(1+\theta\varphi(u)v\varphi'(v)) \quad u,v \in [0,1].$$

In the following proposition we derive sufficient conditions for $Z \geq 0$.

Propostion 6 *Let φ be differentiable and non-negative and monotonly decreasing or non-positive and monotonly increasing on $[0,1]$.*

If

1. *$\varphi(1) = 0$ and*
2. *$|u\varphi'(u)| \leq 1$, $u \in [0,1]$,*

then

$$C(u,v;\theta) = uve^{\theta\varphi(u)\varphi(v)} \quad u,v \in [0,1]$$

is a copula for $\theta \in [-1,0]$.

Proof: We only consider cases of non-negative and monotonely decreasing φ. The complementary case can be shown analogously.

Let $\theta < 0$. We have

$$1 + \theta u\varphi'(u)\varphi(v) \geq 1 \quad u,v \in [0,1]$$

and

$$1 + \theta v\varphi'(v)\varphi(u) \geq 1 \quad u,v \in [0,1].$$

With the condition $u\varphi'(u) \geq -1$, $u \in [0,1]$ we obtain

$$-1 \leq \theta u\varphi'(u)v\varphi'(v) \leq 0 \quad u,v \in [0,1]$$

and therefore,

$$Z = \theta u\varphi'(u)v\varphi'(v) + (1+\theta u\varphi'(u)\varphi(v))(1+\theta v\varphi'(v)\varphi(u)) \geq 0$$

for $u,v \in [0,1]$. □

Example 12 *For* $\varphi(u) = 1 - u^k$*,* $u \in [0,1]$*,* $0 \le k \le 1$ *we have*

$$u\varphi'(u) = -ku^k \ge -1 \;\; u \in [0,1],$$

Hence,

$$C(u,v;\theta) = uve^{\theta(1-u^k)(1-v^k)} \;\; u,v \in [0,1]$$

is a copula for $\theta \in [-1,1]$ *and* $0 < k \le 1$*. For* $k = 1$ *this familiy includes the new copula introduced by Cuadras (2009).*

Example 13 *Consider* $\varphi(u) = e^{-u} - e^{-1}$*,* $u \in [0,1]$*. Then* $|\varphi(u)| \le 1$*,* $u \in [0,1]$ *and*

$$u\varphi'(u) = -ue^{-u} \ge -1 \;\; u \in [0,1].$$

Therefore,

$$C(u,v;\theta) = uv\exp\big(\theta(e^{-u} - e^{-1})(e^{-v} - e^{-1})\big) \;\; u,v \in [0,1]$$

is a copula for $\theta \in [-1,1]$*.*

Example 14 *The function* $\varphi(u) = \ln u$*,* $u \in [0,1]$ *is indeed non-positive and increasing on* $[0,1]$ *but unbounded. He have*

$$u\varphi'(u) = 1 \;\; u \in [0,1],$$

such that

$$C(u,v;\theta) = uve^{\theta \ln u \ln v} \;\; u,v \in [0,1]$$

is a copula for $\theta \in [-1,0]$*. This is the so called Gumbel-Barnett copula which has been studied amongst others by Cuadras (2009).*

Example 15 *For* $\varphi(u) = \min\big(1, \frac{1}{u^k} - 1\big)$*,* $u \in [0,1]$*,* $k > 0$ *we obtain*

$$-1 \le u\varphi'(u) = \begin{cases} 0 & \text{für } u < (1/2)^{1/k} \\ -ku^{-k} & \text{für } u < (1/2)^{1/k} \end{cases}$$

for $\theta \in [-1,1]$*, if* $k \le 1$*. This includes the limiting case* $u\varphi(u) \le \min(u, 1-u)$*,* $u \in [0,1]$ *discussed by Amblard & Girard (2002). In this case*

$$C(u,v;\theta) = uv\exp\left(\theta \min\left(1, \frac{1}{u^k} - 1\right) \min\left(1, \frac{1}{v^k} - 1\right)\right) \;\; u,v \in [0,1]$$

is a copula for $\theta \in [-1,1]$*.*

7 Summary for selected $\varphi(u) = 1 - u^k$

In summary we proved for $\varphi(u) = 1 - u^k$, $u \in [0,1]$ that

$$C(u,v;\gamma,\theta) = uv(1 + \theta\varphi(u)\varphi(v))^{1/\gamma} \;\; u,v \in [0,1]$$

is a copula, if

1. $\theta \in [-1,0)$ and $1/\gamma \in [-1/k, \min(1/k,1)]$ or
2. $\theta \in [0,1]$ and $1/\gamma \in [\max(-1/k,-1), 1/k]$.

1. For $k = 1$ this copula family comprises the FGM and AMH copula for $\theta \in [-1,1]$. Additionally,

   $$C(u,v;\gamma,\theta) = uv(1 + \theta(1-u)(1-v))^{1/\gamma} \;\; u,v \in [0,1]$$

   represents a copula for $\theta \in [-1,1]$ and $-1 \leq 1/\gamma \leq 1$. This holds e.g. for $1/\gamma = -1/2$ which produces the new copula

   $$C(u,v;\gamma=-2,\theta) = \frac{uv}{\sqrt{(1 + \theta(1-u)(1-v))}} \;\; u,v \in [0,1].$$

2. For $k > 1$ the condition $-1/k \leq 1/\gamma \leq 1/k$ is binding for all $\theta \in [-1,1]$. Hence

   $$C(u,v;\gamma=-2,\theta) = \frac{uv}{\sqrt{(1 + \theta(1-u^2)(1-v^2))}} \;\; u,v \in [0,1]$$

   is a copula for all $\theta \in [-1,1]$.

3. For $k < 1$ we have to choose $-1 \leq 1/\gamma \leq 1$. Taking $k = 1/2$ and $\gamma = -2$

   $$C(u,v;\gamma=-2,\theta) = \frac{uv}{\sqrt{(1 + \theta(1-\sqrt{u})(1-\sqrt{v}))}} \;\; u,v \in [0,1]$$

   is a copula for all $\theta \in [-1,1]$.

4. A corresponding result that for $k = 1/2$ and $\gamma = -1/2$ the function

   $$C(u,v;\gamma=-1/2,\theta) = \frac{uv}{(1 + \theta(1-\sqrt{u})(1-\sqrt{v}))^2} \;\; u,v \in [0,1]$$

   is a copula, cannot be achieved using the conditions proved here, because we only have derived necessary conditions.

Finally,

$$C(u,v;\theta) = uv\exp\left(1+\theta(1-u^k)(1-v^k)\right) \;\; u,v\in[0,1]$$

is a copula for $\theta \in [-1,1]$ and $0 < k \leq 1$.

Moreover, these copula families are closed under the construction of weighted power means:

1. In the case $k = 1$

$$\begin{aligned}&\left(\alpha C(u,v;\gamma=-2,\theta_1)^{-2} + (1-\alpha)C(u,v;\gamma=-2,\theta_2)^{-2}\right)^{-1/2}\\ &= \frac{uv}{\sqrt{1+\alpha(\theta_1+(1-\alpha)\theta_2)(1-u)(1-v)}}\end{aligned}$$

for $u,v \in [0,1]$ and $\theta_i \in [-1,1]$, $i = 1,2$ is a copula with dependence parameter $\alpha\theta_1 + (1-\alpha)\theta_2$.

2. Using the weighted geometric mean for $k = 1$

$$C(u,v;\theta_1)^{\alpha}C(u,v;\theta_2)^{1-\alpha} = \exp\left((\alpha\theta_1+(1-\alpha)\theta_2)(1-u)(1-v)\right)$$

for $u,v \in [0,1]$ and $\theta_i \in [-1,1]$, $i = 1,2$ is a copula with dependence parameter $\alpha\theta_1 + (1-\alpha)\theta_2$ as well.

8 Dependence measures

We restrict ourselves to the discussion of Spearman's rank correlation coefficient ρ that takes the form

$$\rho = 12\int_0^1\int_0^1 C(u,v)\mathrm{d}u\mathrm{d}v - 3$$

for a copula $C(u,v)$ (see e.g. Nelsen (1999)).

For certain settings of $\gamma \neq 0$, such that

$$C(u,v;\theta,\gamma) = uv(1+\theta\varphi(u)\varphi(v))^{1/\gamma} \tag{42}$$

is a copula for $\theta \in [-1,1]$, Spearman's ρ can be stated explicitely.

Thus, for $\gamma = 1$ and the copula suggested by Amblard & Girard (2002,2003)

$$C(u,v;\gamma=1,\theta) = uv + \theta u\varphi(u)v\varphi(v) \;\; u,v\in[0,1] \tag{43}$$

for $\theta \in [-1,1]$ we obtain for the rank correlation coefficient

$$\rho = 12\theta \left(\int_0^1 u\varphi(u)\mathrm{d}u \right)^2 .$$

Apparently, the dependence parameter θ and the choice of φ particularly affect the amount of ρ. For a strictly positive or strictly negative φ Spearman's ρ is the larger, the larger the area between φ and the x-axis is.

Example 16 *Regarding $\varphi(u) = 1 - u^k$, $u \in [0,1]$, $k > 0$ we have*

$$\rho = 12\theta \left(\frac{1}{2} - \frac{1}{k+2} \right)^2 = 12\theta \frac{k^2}{4(k+2)^2}$$

for $\theta \in [-1,1]$.

We observe that the area between the graph of $\varphi(u)$ and the x-axis decreases for large k. Therefore, for a given $\theta \in [-1,1]$, the absolute value of Spearman's ρ also decreases. In dependence of k Spearman's ρ probably can only take values in a very small range of $[-1,1]$ as the following tables shows:

k	$[\min(\rho), \max(\rho)]$
0.1	$[-0.00680, 0.00680]$
0.2	$[-0.0248, 0.0248]$
0.3	$[-0.0510, 0.0510]$
0.4	$[-0.0833, 0.0833]$
0.5	$[-0.120, 0.120]$
0.6	$[-0.160, 0.160]$
0.7	$[-0.202, 0.202]$
0.8	$[-0.245, 0.245]$
0.9	$[-0.289, 0.289]$
1.0	$[-0.333, 0.333]$

Table 1: Range of Spearman's ρ for $\varphi(u) = 1 - u^k$, $\gamma = 1$.
and various values of k.

Example 17 *For $\varphi(u) = e^{-u} - e^{-1}$, $u \in (0,1]$ Spearman's ρ rewrites as*

$$\rho = 12\theta \left(\int_0^1 (ue^{-u} - ue^{-1})\mathrm{d}u \right)^2 = 12\theta \left(1 - \frac{5}{2}e^{-1} \right)^2 .$$

For $\theta \in [-1,1]$, Spearman's ρ can only take values in $[-0.0774, 0.0774]$ so that the ability of modelling monotone dependencies is very limited in this setting.

In contrast, for $\gamma \neq 1$ Spearman's ρ can only be determined numerically. We have

$$\rho = 12 \int_0^1 \int_0^1 C(u,v;\theta,\gamma)\mathrm{d}u\mathrm{d}v - 3$$

for $C(.,.;\theta,\gamma)$ according to (42).

Example 18 *Regarding again $\varphi(u) = 1 - u^k$, $u \in [0,1]$, $k > 0$, we obtain with $1/\gamma = -1$ the following ranges for ρ subject to various $k > 0$:*

k	$[\min(\rho), \max(\rho)]$
0.1	$[-0.00675, 0.00687]$
0.2	$[-0.0242, 0.0255]$
0.3	$[-0.0486, 0.0541]$
0.4	$[-0.0774, 0.0914]$
0.5	$[-0.109, 0.137]$
0.6	$[-0.141, 0.190]$
0.7	$[-0.174, 0.251]$
0.8	$[-0.207, 0.319]$
0.9	$[-0.240, 0.395]$
1.0	$[-0.271, 0.479]$

Table 2: Range of Spearman's ρ for $\varphi(u) = 1 - u^k$, $\gamma = -1$.
and various values of k.

The AMH copula for $k = 1$ admits the largest range for Spearman's ρ. Futhermore this copula family can rather capture positive than negative dependence.

In order to study the influence of γ on the range of Spearman's ρ, we choose $\varphi(u) = 1 - u$, $u \in [0,1]$ and vary $1/\gamma$ in the admissible interval $[-1,1]$.

Example 19 *We learn from table 3, that ρ can only capture a very small range of possible dependencies if the absolute value of γ is large.*

γ	$[\min(\rho), \max(\rho)]$
-10	$[-0.0297, 0.0399]$
-8	$[-0.0370, 0.0501]$
-6	$[-0.0491, 0.0673]$
-4	$[-0.0730, 0.102]$
-2	$[-0.142, 0.215]$
-1	$[-0.271, 0.479]$
1	$[-0.333, 0.333]$
2	$[-0.180, 0.158]$
4	$[-0.0938, 0.0769]$
6	$[-0.0634, 0.0508]$
8	$[-0.0479, 0.0379]$
10	$[-0.0385, 0.0303]$

Tabelle 3: Range of Spearman's ρ for $\varphi(u) = 1 - u$ and various values of γ.

9 Tail dependence

9.1 Definitions

According to Ledford & Tawn (1996) we consider the following four asymptotic tail indices, which quantify the relation between U and V in the extreme cases $(U > u, V > u)$ for large $u \in [0,1]$ and $(U < u, V < u)$ for small $u \in [0,1]$, respectively. These indices are only dependent on the copula function $C(.,.)$:

1. Upper (strong) tail coefficient:
$$\lambda_U = \lim_{u \to 1^-} P(U > u | V > u) = \lim_{u \to 1^-} \frac{1 - 2u + C(u,u)}{1 - u}. \tag{44}$$
2. Lower (strong) tail coefficient:
$$\lambda_L = \lim_{u \to 0^+} P(U < u | V < u) = \lim_{u \to 0} \frac{C(u,u)}{u}. \tag{45}$$
3. Upper (weak) tail coefficient:
$$\overline{\chi}_U \equiv \lim_{u \to 1} \frac{P(U > u)P(V > u)}{P(U > u, V > u)} - 1 = \lim_{u \to 1} \frac{(1-u)^2}{1 - 2u + C(u,u)} - 1 \in [-1, 1] \tag{46}$$

4. Lower (weak) tail coefficient:

$$\overline{\chi}_L = \lim_{u \to 0} \frac{P(U < u)P(V < u)}{P(U < u, V < u)} - 1 = \lim_{u \to 0} \frac{u^2}{C(u,u)} - 1 \in [-1, 1] \tag{47}$$

For differentiable $C(u, v)$ the upper (strong) tail coefficient is given by

$$\lambda_U = 2 - \lim_{u \to 1^-} \frac{dC(u,u)}{du}$$

and the weak by

$$\lambda_L = \lim_{u \to 0^+} \frac{dC(u,u)}{du}.$$

To obtain formulas for the weak tail coefficients we transform U and V to the so-called uniform Fréchet marginal distribution via the transformation

$$S = -1/\log U \text{ and } T = -1/\log V,$$

such that

$$P(S > s) = P(T > s) = P(U > e^{-1/s}) = 1 - e^{-1/s}$$

holds for $s \geq 0$.

A function $\mathcal{L}$ is said to be slowly varying at infinity, if for all $c > 0$

$$\frac{\mathcal{L}(ct)}{\mathcal{L}(t)} \to 1 \text{ for } t \to \infty.$$

If

$$\begin{aligned} F_U(t,t) &:= P(S > t, T > t) = P\left(U > e^{-1/t}, V > e^{-1/t}\right) \\ &= 1 - 2e^{-1/t} + C(e^{-1/t}, e^{-1/t}) \\ &\approx \mathcal{L}(t) \left(\frac{1}{t}\right)^{1/\eta} \quad \text{for large } t \end{aligned}$$

holds for $\eta \in [0, 1]$ and a slowly varying function $\mathcal{L}(t)$ with limit c for $t \to \infty$, the upper (weak) tail coefficient can be written as

$$\overline{\chi}_U = 2\eta - 1. \tag{48}$$

If on the other hand

$$\begin{aligned} F_L(t,t) &= P(S < t, T < t) = P\left(U < 1 - e^{-1/t}, V < 1 - e^{-1/t}\right) \\ &= C\left(1 - e^{-1/t}, e^{-1/t}\right) \\ &\approx \mathcal{L}(t) \left(\frac{1}{t}\right)^{1/\eta} \quad \text{for large } t, \end{aligned}$$

the lower (weak) tail coefficient is

$$\overline{\chi}_L = 2\eta - 1. \tag{49}$$

Note that $\lambda_U = 0$ ($\lambda_L = 0$) if $\overline{\chi}_U < 1$ ($\overline{\chi}_L < 1$). Only for $\overline{\chi}_U = 1$ ($\overline{\chi}_L = 1$), the strong tail coefficients λ_U (λ_L) can take values $c \neq 0$, where $c = \lim_{t\to\infty} \mathcal{L}(t)$.

9.2 Tail dependence for $\gamma > 0$

Propostion 7 *Let φ be differentiable with $\varphi(1) = 0$ and let*

$$C(u, v; \gamma, \theta) = uv\left(1 + \theta\varphi(u)\varphi(v)\right)^{1/\gamma} \quad u, v \in [0, 1]$$

be a copula for appropriate $\gamma > 0$ and $\theta \in [-1, 1]$.

Then:

1. $\lambda_L = \lambda_U = 0$.
2. $\overline{\chi}_L = 0$ *with*

$$\begin{aligned} F_L(t,t) &= C\left(1 - e^{-1/t}, e^{-1/t}, \gamma, \theta\right) \\ &\approx \left(1 + \frac{\theta}{\gamma}\varphi'(1)^2\right)\frac{1}{t^2} \;\; \textit{for large } t. \end{aligned}$$

3. $\overline{\chi}_U = 0$ *with*

$$\begin{aligned} F_U(t,t) &= 1 - 2e^{-1/t} + C(e^{-1/t}, e^{-1/t}) \\ &\approx \left(1 + \frac{\theta}{\gamma}\varphi'(1)^2\right)\frac{1}{t^2} \;\; \textit{for large } t. \end{aligned}$$

Proof:

1. Due to $\varphi(1) = 0$,

$$\begin{aligned} \lambda_U &= \lim_{u\to 1^-}\left(2 - \frac{dC(u,u;\gamma,\theta)}{du}\right) \\ &= \lim_{u\to 1^-}\left(2 - 2u(1+\theta\varphi(u)^2)^{1/\gamma} - u^2\frac{2\theta}{\gamma}(1+\theta\varphi(u)^2)^{1/\gamma-1}\varphi(u)\varphi'(u)\right) \\ &= 2 - 2 = 0. \end{aligned}$$

$\varphi(u)$ and $\varphi'(u)$ are bounded on $[0,1]$. Hence,

$$\lambda_L = \lim_{u \to 0^+} \frac{dC(u,u;\gamma,\theta)}{du} = 0$$

can be shown analogously.

2. The Taylor expansion $x^{1/\gamma} \approx 1 + 1/\gamma(x-1)$ provides

$$\begin{aligned} F_L(t,t) &= C\left(1-e^{-1/t}, 1-e^{-1/t}, \gamma, \theta\right) \\ &\approx \left(1-e^{-1/t}\right)^2 \left(1 + \frac{\theta}{\gamma}\varphi\left(1-e^{-1/t}\right)^2\right) \quad \text{for large } t. \end{aligned}$$

Using the approximations $\varphi(u) \approx \varphi(1) + \varphi'(1)(u-1)$ and $1 - e^{-1/t} \approx -1/t$, $e^{-2/t} \approx 1 - 2/t$ we finally obtain

$$\begin{aligned} F_L(t,t) &\approx \left(1-e^{-1/t}\right)^2 \left(1 + \frac{1}{\gamma}\left(1+\theta\varphi'(1)^2 e^{-2/t} - 1\right)\right) \\ &\approx \frac{1}{t^2}\left(1 + \frac{\theta}{\gamma}\varphi'(1)^2\left(1-\frac{2}{t}\right)\right) \\ &\approx \left(1 + \frac{\theta}{\gamma}\varphi'(1)^2\right)\frac{1}{t^2} \quad \text{for large } t. \end{aligned}$$

Hence, $\eta = 1/2$, $\overline{\chi}_L = 0$ and

$$\mathcal{L}(t) = \left(1 + \frac{\theta}{\gamma}\varphi'(1)^2\right).$$

3. With $x^{1/\gamma} \approx 1 + 1/\gamma(x-1)$ we have

$$\begin{aligned} F_U(t,t) &= 1 - 2e^{-1/t} + C(e^{-1/t}, e^{-1/t}) \\ &\approx 1 - 2e^{-1/t} + e^{-2/t}\left(1 + \frac{1}{\gamma}\left(1 + \theta\varphi\left(e^{-1/t}\right)^2 - 1\right)\right) \\ &\approx 1 - 2e^{-1/t} + e^{-2/t}\left(1 + \frac{\theta}{\gamma}\varphi'(1)^2\left(1-e^{-1/t}\right)^2\right) \\ &\approx \left(1-e^{-1/t}\right)^2\left(1 + \frac{\theta}{\gamma}\varphi'(1)^2 e^{-2/t}\right) \\ &\approx \left(1 + \frac{\theta}{\gamma}\varphi'(1)^2\left(1-\frac{2}{t}\right)\right)\frac{1}{t^2} \\ &\approx \left(1 + \frac{\theta}{\gamma}\varphi'(1)^2\right)\frac{1}{t^2} \quad \text{for large } t. \end{aligned}$$

Thus, $\eta = 1/2$, $\overline{\chi}_U = 0$ and

$$\mathcal{L}(t) = \left(1 + \frac{\theta}{\gamma}\varphi'(1)^2\right). \quad \square$$

Example 20 *If C is a FGM copula, then $\gamma = 1$ and $\varphi(u) = 1 - u$ for $u \in [0,1]$. Using $\varphi'(1) = -1$ we get*

$$F_L(t,t) \approx (1+\theta)\frac{1}{t^2}.$$

This corresponds to the result obtained by Currie (1999), p. 11. In addition,

$$F_U(t,t) \approx (1+\theta)\frac{1}{t^2}$$

also agrees with Currie (1999), p. 10.

9.3 Tail dependence for $\gamma < 0$

Propostion 8 *Let φ be differentiable with $\varphi(1) = 0$ and let*

$$C(u,v;\gamma,\theta) = uv\left(1 + \theta\varphi(u)\varphi(v)\right) \quad u,v \in [0,1]$$

be a copula for appropriate $\gamma < 0$ and $\theta \in [-1,1]$.

Then

1. $\lambda_L = \lambda_U = 0$.
2. $\overline{\chi}_L = 0$ *with*

$$\begin{aligned} F_L(t,t) &= C\left(1 - e^{-1/t}, 1 - e^{-1/t}, \gamma, \theta\right) \\ &\approx \left(1 - \frac{\theta}{|\gamma|}\varphi'(1)^2\right)^{-1} \frac{1}{t^2} \quad \text{for large } t. \end{aligned}$$

3. $\overline{\chi}_U = 0$ *with*

$$\begin{aligned} F_U(t,t) &= 1 - 2e^{-1/t} + C(e^{-1/t}, e^{-1/t}) \\ &\approx \left(1 + \frac{\theta}{|\gamma|}\varphi'(1)^2\right) \frac{1}{t^2} \quad \text{for large } t. \end{aligned}$$

Proof: We consider

$$C(u,v;\gamma,\theta) = \frac{uv}{(1 + \theta\varphi(u)\varphi(v))^{1/|\gamma|}} \quad u,v \in [0,1].$$

1. This is an immediate conclusion of part 2 and 3, because $\chi_L = \chi_U = 0 < 1$ implies $\lambda_U = \lambda_L = 0$.

2. Using $x^{1/|\gamma|} \approx 1 + 1/|\gamma|(x-1)$ we get

$$\begin{aligned} F_L(t,t) &= C\left(1-e^{-1/t}, 1-e^{-1/t}, \gamma, \theta\right) \\ &\approx \left(1-e^{-1/t}\right)^2 \frac{1}{\left(1+\frac{\theta}{|\gamma|}\varphi'(1)^2\left(e^{-1/t}\right)^2\right)}. \end{aligned}$$

The expansions

$$\begin{aligned} (1-e^{-1/t})^2 &= 1-2e^{-1/t}+e^{-2/t} \approx 1-2\left(1-\frac{1}{t}+\frac{1}{2t^2}-\frac{1}{6t^3}+\frac{1}{24t^2}\right) \\ &+1-\frac{2}{t}+\frac{4}{2t^2}-\frac{8}{t^3}+\frac{16}{24t^4} = \frac{1}{t^2}-\frac{1}{t^3}+\frac{14}{24t^4} \end{aligned}$$

and

$$\left(e^{-1/t}\right)^2 = e^{-2/t} = 1-\frac{2}{t}+\frac{4}{2t^2}$$

lead to

$$\begin{aligned} F_L(t,t) &\approx \frac{1}{t^2}\frac{1-1/t+14/(24t^2)}{1+\frac{\theta}{|\gamma|}\varphi'(1)^2\left(1-2/t+2/t^2\right)} \\ &\approx \frac{1}{1+\frac{\theta}{|\gamma|}\varphi'(1)^2}\frac{1}{t^2} \quad \text{for large } t. \end{aligned}$$

Hence, $\eta = 1/2$, $\overline{\chi}_L = 0$ and

$$\mathcal{L}(t) = \left(1+\frac{\theta}{|\gamma|}\varphi'(1)^2\right)^{-1}.$$

3. The Taylor expansion $x^{-1/|\gamma|} \approx 1-\frac{1}{\gamma}x$ leads to

$$\begin{aligned} F_U(t,t) &= 1-2e^{-1/t}+C(e^{-1/t},e^{-1/t}) \\ &\approx 1-2e^{-1/t}+e^{-2/t}\left(1-\frac{\theta}{|\gamma|}\varphi(e^{-1/t})^2\right) \\ &\approx (1-e^{-1/t})^2 - e^{-2/t}\frac{\theta}{|\gamma|}\varphi'(1)^2\left(e^{-1/t}-1\right)^2 \\ &\approx (1-e^{-1/t})^2 - e^{-2/t}\frac{\theta}{|\gamma|}\varphi'(1)^2\left(1-e^{-1/t}\right)^2 \\ &\approx \left(1-\frac{\theta}{|\gamma|}\varphi'(1)^2(1-2/t)\right)\left(1-e^{-1/t}\right)^2 \\ &\approx \left(1-\frac{\theta}{|\gamma|}\varphi'(1)^2\right)\frac{1}{t^2} \quad \text{for large } t. \end{aligned}$$

Therefore, $\eta = 1/2$, $\overline{\chi}_U = 0$ and

$$\mathcal{L}(t) = \left(1-\frac{\theta}{|\gamma|}\varphi'(1)^2\right).\square$$

Example 21 *Let C be the AMH copula, then $\gamma = -1$ and $\varphi(u) = 1 - u$, $u \in [0, 1]$. With $\varphi'(1) = -1$ and*

$$F_L(t,t) \approx \frac{1}{1+\theta}\frac{1}{t^2}$$

we obtain the result of Currie (1999), p. 5. Furthermore,

$$F_U(t,t) \approx (1-\theta)\frac{1}{t^2}$$

also agrees with Currie (1999).

9.4 Tail dependence for $\gamma = 0$

Propostion 9 *Let φ be differentiable with $\varphi(1) = 0$ and let*

$$C(u, v; \theta) = uv \exp(\theta\varphi(u)\varphi(v)) \;\; u, v \in [0, 1]$$

be a copula for $\theta \in \Theta \subseteq [-1, 1]$.

Then

1. $\lambda_L = \lambda_U = 0$.
2. $\overline{\chi}_L = 0$ *with*

$$\begin{aligned} F_L(t,t) &= C\left(1 - e^{-1/t}, 1 - e^{-1/t}, \gamma, \theta\right) \\ &\approx \left(1 - \theta\varphi'(1)^2\right)\frac{1}{t^2} \;\; \textit{for large } t. \end{aligned}$$

3. $\overline{\chi}_U = 0$ *with*

$$\begin{aligned} F_U(t,t) &= 1 - 2e^{-1/t} + C(e^{-1/t}, e^{-1/t}) \\ &\approx \left(1 - \theta)\varphi'(1)^2\right)\frac{1}{t^2} \;\; \textit{for large } t. \end{aligned}$$

Proof:

1. Again, $\lambda_U = \lambda_L = 0$ results from $\overline{\chi}_L = \overline{\chi}_U = 0 < 1$ proved in the second and third part.

2. Due to

$$\begin{aligned}
F_L(t,t) &= C\left(1-e^{-1/t}, 1-e^{-1/t}, \theta\right) \\
&= \left(1-e^{-1/t}\right)^2 \exp\left(\theta\varphi\left(1-e^{-1/t}\right)^2\right) \\
&\approx \left(1-e^{-1/t}\right)^2 \left(1+\theta\varphi\left(1-e^{-1/t}\right)^2\right) \\
&\approx \left(1-e^{-1/t}\right)^2 \left(1+\theta\varphi'(1)^2\left(e^{-1/t}\right)^2\right) \\
&\approx \frac{1}{t^2}\left(1+\theta\varphi'(1)^2\left(1-\frac{2}{t}\right)\right) \\
&\approx \left(1+\theta\varphi'(1)^2\right)\frac{1}{t^2} \text{ for large } t,
\end{aligned}$$

we obtain $\eta = 1/2$, $\overline{\chi}_L = 0$ and

$$\mathcal{L}(t) = 1 + \theta\varphi'(1)^2.$$

3. From

$$\begin{aligned}
F_U(t,t) &= 1-2e^{-1/t} + C\left(e^{-1/t}, e^{-1/t}, \theta\right) \\
&= 1-2e^{-1/t} + e^{-2/t}\exp\left(\theta\varphi\left(e^{-1/t}\right)^2\right) \\
&\approx 1-2e^{-1/t} + e^{-2/t}\left(1+\theta\varphi\left(e^{-1/t}\right)^2\right) \\
&\approx 1-2e^{-1/t} + e^{-2/t}\left(1+\theta\varphi'(1)^2\left(e^{-1/t}-1\right)^2\right) \\
&\approx (1-e-1/t)^2\left(1+\theta\varphi'(1)^2 e^{-2/t}\right) \\
&\approx \left(1+\theta\varphi'(1)^2\right)\frac{1}{t^2} \text{ for large } t
\end{aligned}$$

we get $\eta = 1/2$, $\overline{\chi}_U = 0$ and

$$\mathcal{L}(t) = 1 + \theta\varphi'(1)^2.$$

Example 22 *The BG copula for $\theta \in [-1, 0]$ takes the form*

$$C(u,v;\theta) = uve^{\theta \ln u \ln v} \quad u, v \in [0,1].$$

with $\varphi(u) = \ln u$, $u \in [0,1]$ and $\varphi'(1) = 1$. Hence,

$$\mathcal{L}(t) = 1 + \theta.$$

Example 23 *Cuadras (2009) investigates the function $\varphi(u) = 1 - u$, $u \in [0,1]$, which leads to*

$$\mathcal{L}(t) = 1 + \theta.$$

10 Conclusion

We have identified sufficient and partly necessary conditions for a family of aggregation functions that is closed under the construction of weighted power means to be a copula. These copula families generalize results given in literature for the FGM, AMH and BG copula. Furthermore we have nvestigated for theses families the amount of Spearman's ρ as an widely-used dependence measure. Thereby it has arisen that the ranges are often considerably smaller than those for the FGM and AMH copula. These two copula families and our generalizations have in common that they can't capture tail dependence. This holds both for the strong and for the weak tail coefficient.

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