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# A Tail Quantile Approximation Formula for the Student t and the Symmetric Generalized Hyperbolic Distribution

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Abstract

Calculating a large number of tail probabilities or tail quantiles for a given distribution families becomes very challenging, if both the cumulative and the inverse distribution function are not available in closed form. In case of the Gaussian and Student t distribution, quantile approximations are already available. This is not the case for the (symmetric) generalized hyperbolic distribution (GHD) whose popularity steadily increases and which includes both Gaussian and Student t as limiting case. Within this paper we close this gap and derive one possible tail approximation formula for the GHD as well as for the Student t distribution.

*Keywords and phrases:* Generalized hyperbolic distribution; Quantile approximation; Student t distribution.

## 1 Introduction

Quantiles of probability distributions are basic building blocks of various risk measures in finance like, for example, the Value-at-Risk or the Expected Shortfall (see McNeil et al., 2005). Calculating them (or many of them) might be very cumbersome if both the inverse cumulative and the cumulative distribution function are not available in closed form. In this case analytic approximations are needed. An approach applicable to all distributions was derived by Cornish & Fisher (1937) using the distribution function's moments. More accuracy offer quantile formulas especially developed for certain distributions. There are already approximations for the Gaussian quantile (see Reiss, 1989) or the Student t quantile (see Gafer & Kafader, 1984, which propose a formula for the distribution's centre), but – to our best knowledge – there is no approximation formula for a popular super-model, the symmetric generalized hyperbolic distribution, although this distribution family is widely applied in statistics (see e.g. McNeil et al., 2005, Eberlein & Keller, 1995, Prause, 1999).

Within this paper we sketch the well-known tail quantile approximation of the Gaussian distribution and derive formulas for the Student t and the hyperbolic distribution. For this purpose, the paper is structured as follows: first, we fix notation and give some definitions. Second, general results on the generalized hyperbolic distributions are briefly summarized. Third, the quantile approximation formulas are derived in Section 4.

## 2 Notation and definitions

Let us fix notation first. For a given random variable $X$, $f$ denotes the probability density function, $F$ the cumulative distribution function and $F^{-1}$ the corresponding quantile function. We use $\varphi$ and $\Phi$ for the standardized Gaussian density and distribution function, respectively. Furthermore, $X \sim \Phi$ means that $X$ is normally distributed. The symbol $\simeq$ denotes asymptotic equivalence, whereas approximations are characterized by $\approx$. Later on, we approximate the standard Gaussian tail by Mill's ratio (see Ruben, 1963), which reads as $1 - \Phi(x) \simeq \varphi(x)/x$. The Landau symbol "little **o**" indicates, that one (real-valued) function asymptotically dominates another, i.e. $f = \mathbf{o}(g)$ means that $\lim_{x\to a} f(x)/g(x) = 0$ if $a$ denotes the right endpoint of the domain of $f$ and $g$. This relation is invariant under monotone transformations (see Hardy & Wright, 1979). We also use the modified Bessel function of the third kind (see Abramovitz & Stegun, 1965) which we briefly refer to as Bessel function in the sequel. Its formula reads as

$$K_\lambda(x) = \frac{1}{2}\int_0^\infty t^{\lambda-1} e^{-\frac{1}{2}x(t+t^{-1})}dt$$

for $\lambda \in \mathbb{R}$ and $x > 0$. Following Abramowitz & Stegun (1965), for $x \to \infty$ and $\mu = 4\lambda^2$, we obtain the approximation

$$K_\lambda(x) \approx \sqrt{\frac{\pi}{2x}}e^{-x}\left(1 + \frac{\mu-1}{8x} + \frac{(\mu-1)(\mu-9)}{2!(8x)^2} + \dots\right) = \sqrt{\frac{\pi}{2x}}e^{-x} + \epsilon. \tag{2.1}$$

The error term converges to zero with $x \to \infty$, but the speed decreases with increasing $\lambda$. The Bessel function is part of the normalizing constant of the Generalized Inverse Gaussian (GIG) distribution on $\mathbb{R}_+$ whose density reads as

$$gig(x;\lambda,\chi,\psi) = \underbrace{\frac{\left(\frac{\psi}{\chi}\right)^{\frac{\lambda}{2}}}{2\cdot K_\lambda(\sqrt{\chi\psi})}}_{c(\lambda,\chi,\psi)} \cdot x^{\lambda-1}\cdot\exp\left(-\frac{1}{2}(\chi x^{-1}+\psi x)\right)\cdot \mathbf{1}_{(0,\infty)}(x), \tag{2.2}$$

where the admissible domain for the parameters is given by $\chi > 0,\ \psi \geq 0$ if $\lambda < 0$, $\chi > 0,\ \psi > 0$ if $\lambda = 0$ and $\chi \geq 0,\ \psi > 0$ if $\lambda > 0$ (see, for instance, Jørgensen, 1982). It can be shown for $c(\lambda,\chi,\psi)$ that

$$\lim_{\chi\to 0} c(\lambda,\chi,\psi) = \frac{\left(\frac{\psi}{2}\right)^{\lambda}}{\Gamma(\lambda)}, \quad \lim_{\psi\to 0} c(\lambda,\chi,\psi) = \frac{\left(\frac{\chi}{2}\right)^{-\lambda}}{\Gamma(-\lambda)}. \tag{2.3}$$

If both $\chi,\psi\to\infty$ and $\sqrt{\chi/\psi}\to\omega$, the GIG's density converges to the Dirac measure in $\omega$, denoted by $\delta_\omega$ (see Barndorff-Nielsen, 1978). An important concept in the derivation of our formulas are so-called *variance-mixture of normals* or *normal-variance mixtures* (see Barndorff-Nielsen, 1977), defined as

$$X|Z = z \sim \mathcal{N}(\mu, z\sigma^2) \text{ with } z\in[0,\infty)\,, \text{ and } Z\sim M, \tag{2.4}$$

where $\mathcal{N}$ denotes the normal distribution and $M$ is an arbitrary *mixing* distribution on the positive axis. Consequently, the corresponding mixture density derives as

$$f(x) = \int_0^\infty n(x;\mu,z\sigma^2)\cdot m(z)dz. \tag{2.5}$$

where $n$ and $m$ denote the Gaussian density and the density of $M$, respectively (provided the existence).

# 3 The symmetric generalized hyperbolic family

The *Generalized Hyperbolic Distribution* (GHD) has been introduced by Barndorff-Nielsen (1977) as a model for the size of beans. But it does also reasonably well in modelling financial data (see Eberlein & Keller, 1995 or Prause, 1999). It is constructed as a specific normal mean-variance mixture using the GIG from (2.2) as mixture distribution (see Blæsild & Jensen, 1980). Forth on, we restrict to the symmetric case. Setting $\chi=\delta^2,\psi=\alpha^2$ for $\alpha>0,\ \delta>0$ in (2.2) and applying formula (2.4), the symmetric GH-density is

$$h(x;\lambda,\alpha,\delta,\mu,\sigma^2) \quad = \quad \frac{\left(\frac{\alpha}{\delta}\right)^{\lambda}}{\sqrt{2\pi}K_{\lambda}(\alpha\delta)} \frac{K_{\lambda-\frac{1}{2}}\left(\alpha\sqrt{\delta^2+\left(\frac{x-\mu}{\sigma}\right)^2}\right)}{\left(\sqrt{\delta^2+\left(\frac{x-\mu}{\sigma}\right)^2}/\alpha\right)^{\frac{1}{2}-\lambda}}. \tag{3.1}$$

For alternative parameterizations we refer to McNeil et al. (2005). All moments as well as the moment generating function of the GHD do exist. Above that, the GHD is infinitely divisibile (Barndoff-Nielsen et al., 1982). There is also a multivariate elliptical GHD available which is closed under margining and conditioning. The GHD allows to rebuild flexible shape behaviour. In particular the tail heaviness varies from light via semi-heavy to heavy, at least if the limit cases are included. Heavy tails are obtained, for example, by the Student t distribution which arises in the limit if $\alpha \to 0, \delta = \sqrt{\nu}$ and $\lambda = -\nu/2$ (see Blæsild, 1999): Setting $\mu = 0$ and $\sigma = 1$ for reason of simplicity,

$$\begin{aligned} h(x,-\nu/2,0,\sqrt{\nu},0,1) &= \lim_{\alpha\to 0}\int_0^{\infty} n(x;0,\omega)\cdot gig(\omega;-\nu/2,\nu,\alpha^2)d\omega \\ &= \frac{1}{\sqrt{2\pi}}\cdot\frac{(\nu/2)^{\nu/2}}{\Gamma(\nu/2)}\int_0^{\infty}\omega^{-\nu/2-1/2-1}\exp\left(-\frac{\nu+x^2}{2\omega}\right)d\omega \\ &= \frac{1}{\sqrt{2\pi}}\cdot\frac{2^{-\nu/2}}{\Gamma(\nu/2)\nu^{-\nu/2}}\cdot\frac{\Gamma(1/2+\nu/2)}{\left(\frac{\nu+x}{2}\right)^{(\nu+1)/2}} \\ &= \frac{\Gamma((\nu+1)/2)}{\sqrt{\pi}\,\Gamma(\nu/2)\sqrt{\nu}}\cdot(1+x^2/\nu)^{-\frac{\nu+1}{2}}. \end{aligned}$$

Semi-heavy tails do occur while the parameters are within their "regular" boundaries. Finally, light tails occur in the limit $\delta, \alpha \to \infty$ while $\delta/\alpha \to \omega \in \mathbb{R}$. In this case, the GIG tends towards the Dirac measure and the Gaussian distribution remains. In the next section we derive approximation formulas for tail quantiles for each of the three cases considered above: The normal distribution (section 4.1), where the formula is already established in the literature, the Student t distribution (4.2) and the symmetric GHD (4.3).

# 4 Tail quantile approximation formulae

The $\alpha$-quantile of a univariate distribution function $F$, often denoted by $q_\alpha = q_\alpha(F)$, is defined as $\inf\{x \in \mathbb{R} : F(X) \geq \alpha\}$ which equals the inverse distribution function in the continuous case. However, the inverse of $F$ is rarely available in closed form except for a few cases, as for example the logistic distribution. A solution to this problem might be to search (analytically or numerically) for the root of $F(x) - \alpha = 0$ which respects the $\alpha$-quantile. But if in addition to that also $F$ is not available in closed form, the density $f$ has (in case of its existence) to be integrated numerically. In toto this requires high computational effort, at least if lots of quantiles are needed.

Within this paper, we focus on analytic approximations for high quantiles of three representants of the GHD family. We present the approach for the Gaussian distribution and modify it in order to construct a high Student t quantile formula. Eventually we derive a quantile approroixmation for the symmetric GHD which is based on the Gamma distribution. In general, finding a quantile approximation meams solving

$$1 - \int_{-\infty}^{x^*} h(u; \lambda, \alpha, \delta, \mu, \sigma^2) du = \frac{1}{t} \tag{4.1}$$

for $x^*$. The result is the $(1 - 1/t)$-quantile, which is a high quantile for large $t$.

## 4.1 An approximation formula for the high Gaussian quantiles

There is a vast amount of tables for the Gaussian quantile. For an overview, see Johnson et al. (1994). Strecok (1986) derives an approximative formulas for the error function which is a modified version of the Gaussian distribution. Within this paper, we compute the quantile for right tail, following an approach briefly sketched in Reiss (1989). Assuming zero mean and unit variance, and using Mill's ratio (4.1) "asymptotically" we can write

$$\begin{aligned} 1 - \Phi(x_G^*) = \frac{1}{t} \quad &\simeq \quad \frac{\varphi(x_G^*)}{x_G^*} = \frac{1}{t} \Longleftrightarrow \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{1}{2} {x_G^*}^2\right) \cdot \frac{1}{x_G^*} = \frac{1}{t} \\ &\Longleftrightarrow \quad \frac{1}{2}\ln(2\pi) + \frac{1}{2}{x_G^*}^2 + \ln x_G^* = \ln t. \end{aligned} \tag{4.2}$$

Note that $x_G^*(t)$ is increasing in $t$, i.e. $t \to \infty$ induces $x_G^* \to \infty$. As $\frac{1}{2}\ln(2\pi) + \ln x_G^* = \mathbf{o}\left({x_G^*}^2\right)$, (4.2) asymptotically simplifies to $\frac{1}{2}{x_G^*}^2 \simeq \ln t$. Consequently,

$$x_G^* = \sqrt{2\ln t} + \epsilon_t, \tag{4.3}$$

where the variable $\epsilon_t$ denotes the error made in ignoring the logarithmic expressions. Per construction, $\epsilon_t = \mathbf{o}(\sqrt{2\ln t})$ holds. Now replacing $x_G^*$ in (4.2) by (4.3),

$$\frac{1}{\sqrt{2\pi}} \cdot \frac{1}{(2\ln t)^{1/2} + \epsilon_t} \cdot \exp\left\{-\frac{1}{2}\left[(2\ln t)^{1/2} + \epsilon_t\right]^2\right\} = \frac{1}{t}.$$

Asymptotically, i.e. for $t \to \infty$, $\epsilon_t$ can be omitted in the denominator of the last formula and we get

$$\frac{1}{\sqrt{2\pi}} \cdot \frac{1}{(2\ln t)^{1/2}} \cdot \exp\left(-\frac{1}{2}\left[2\ln t + 2\epsilon_t(2\ln t)^{1/2} + \epsilon_t^2\right]\right) = \frac{1}{t}.$$

Moreover, we can also ignore $\epsilon_t^2$, because $\epsilon_t = \mathbf{o}\left(\sqrt{2\ln t}\right)$ induces $\epsilon_t^2 = \mathbf{o}\left(2\ln t\right)$. Taking logarithms, the last equation changes to

$$\frac{1}{2}\ln(2\pi) + \frac{1}{2}\ln(2\ln t) + \frac{1}{2}2\ln t + \frac{1}{2}2\epsilon_t(2\ln t)^{1/2} = \ln t.$$

Solving for $\epsilon_t$ finally yields

$$\epsilon_t \simeq -\frac{\ln(2\pi) + \ln(2\ln t)}{2(\sqrt{2\ln t})}.$$

Plugging this formula into (4.3), the final tail quantile approximation is

$$x_G^* \approx (2\ln t)^{1/2} - \frac{1}{2(2\ln t)^{1/2}}\left[\ln(2\pi) + \ln(2\ln t)\right]. \tag{4.4}$$

Eventually, we compare our result (see Table 1 and Figure 4.1) with the implementation from the software package **R** which is based on Wichura (1988) and the results of Owen (1962); both implementations use complex, multi-level algorithms. Figure 4.1 indicates that approximations based on (4.4) are closer to that of the **R** and slightly smaller than that of Owen (1962).

## 4.2 An approximation formula for the high Student t quantiles

As seen before, the Student t distribution arises as limit case of the GHD. Up to now, there already exists some literature about Student t quantiles (cf. Goldberg & Levine, 1946, Peiser, 1943 and Gafer & Kafader, 1984). However, all of them focus on moderate quantiles. To approximate $(1-1/t)$-quantiles for large $t$, we again start with the mixture representation of the GHD: If $T_\nu$ denotes the cumulative distribution function of a Student t distribution with $\nu$ degrees of freedom, the corresponding $(1-1/t)$-quantile $x_T^*$ satisfies

$$1 - T_\nu(x_T^*) = \frac{1}{t} \Longleftrightarrow \int\limits_{x_T^*}^{\infty}\int\limits_{0}^{\infty} n_1(x;0,\omega) gig(\omega;-\nu/2,\nu,0) d\omega dx = \frac{1}{t}$$

$$\Longleftrightarrow \int_0^\infty \left(1 - \Phi\left(\frac{x_T^*}{\sqrt{\omega}}\right)\right) \cdot gig(\omega;-\nu/2,\nu,0) d\omega = \frac{1}{t}. \tag{4.5}$$

Before applying Mill's ratio in (4.5) as in (4.2), we have to check whether the error caused by this approximation tends to zero for $x_T^* \to \infty$. It can be shown by partial integration that(4.5) is equivalent to

$$\int_0^\infty \left(1 - \Phi\left(\frac{\varphi(x_T^*/\sqrt{\omega})}{x_T^*/\sqrt{\omega}} - \int_{x_t^*/\omega}^\infty \frac{1}{v^2}\varphi(v)dv\right)\right) \cdot gig(\omega;-\nu/2,\nu,0) d\omega = \frac{1}{t}. \tag{4.6}$$

Leaving out the second term in the brackets respects Mill's ratio. It can be used if the error $\Delta(x_T^*)$ created by the existence of the second term tends to zero for $x_T^* \to \infty$. This term $\Delta(x_T^*)$ reads as

$$\Delta(x_T^*) = -\int_0^\infty \int_{x_t^*/\omega}^\infty \frac{1}{v^2}\varphi(v)dv \; gig(\omega;\nu/2,\nu,0) d\omega.$$

Using (2.1) we can simplify it to

$$\Delta(x_T^*) \simeq -\frac{(\nu/2)^{\nu/2} 2^{(\nu+1)/2}\Gamma\left(\frac{\nu-1}{2}\right)}{\sqrt{2\pi}\Gamma(\nu/2)} \int_{x_T^*}^\infty (\nu + y^2)^{-\frac{\nu-1}{2}} dy \to 0 \text{ for } x_T^* \to \infty.$$

Due to this result we can use Mill's ratio as follows:

$$\int_0^\infty \varphi\left(\frac{x_T^*}{\sqrt{\omega}}\right) / \frac{x_T^*}{\sqrt{\omega}} \cdot gig(\omega;-\nu/2,\nu,0) d\omega = \frac{1}{t}$$

$$\Longleftrightarrow \quad \frac{1}{\sqrt{2\pi}x_T^*} \int_0^\infty c(-\nu/2,\nu,0)\omega^{-\frac{\nu+1}{2}} \exp\left(-\frac{(x_T^*)^2+\nu}{2\omega}\right) d\omega = \frac{1}{t}, \tag{4.7}$$

which is asymptotically equivalent to (4.5). Now notice that the above integrand corresponds – up to an integration constant – to a GIG density which integrates to one. Enlarging the integrand artificially we get

$$\frac{1}{x_T^*}\frac{1}{\sqrt{2\pi}}\frac{c(-\frac{\nu}{2},\nu,0)}{c(-\frac{\nu}{2}+1/2,\nu+(x_T^*)^2,0)}=\frac{1}{t}.$$

Applying formula (2.3) we obtain

$$\frac{1}{x_T^*}\frac{1}{\sqrt{2\pi}}\frac{(\nu/2)^{\nu/2}}{\Gamma(\nu/2)}\Gamma\left(\frac{\nu}{2}-\frac{1}{2}\right)\left(\frac{\nu+(x_T^*)^2}{2}\right)^{(-\nu/2+1/2)}=\frac{1}{t}.$$

A simple reformulation of the last expression gives

$$\frac{(\nu/2)^{\nu/2}}{\sqrt{2\pi}\Gamma(\nu/2)}\cdot\frac{\Gamma\left(\frac{\nu}{2}-\frac{1}{2}\right)}{2^{(-\nu/2+1/2)}}\cdot\frac{(x_T^*)^{-\nu+1}}{x_T^*}\cdot\left(\underbrace{\frac{\nu}{(x_T^*)^2}}_{\to 0\ \text{for}\ x_T^*\to\infty}+1\right)^{-(\nu-1)/2}=\frac{1}{t}.$$

Finally, solving for $x_T^*$, a very simple tail quantile approximation for the Student t distribution results:

$$x_T^*\approx\sqrt[\nu]{c\cdot t}\quad\text{with}\quad c=\frac{(\nu/2)^{\nu/2}}{\Gamma(\nu/2)\sqrt{2\pi}\cdot 2^{-\nu/2+1/2}}\cdot\Gamma\left(\frac{\nu-1}{2}\right).$$

To demonstrate the accuracy of our formula, we compare it to the corresponding **R** routine, which is based on Hill (1970), and the values of Gafer & Kafader (1984) which uses a modified Gaussian quantile. The results are displayed in Table 4.2 and Figure 4.2. Our formula seems to be applicable especially for quantiles higher than 0.9999. In this case, our quantiles are higher than that of Hill (1970) but lower than that of Gafer & Kafader (1984).

## 4.3 An approximation formula for high GH quantiles

Our approximation is based on the Gamma distribution and restricted to $\lambda > 0$ which covers a broad range of the parameter set.

**Theorem** 1 (GHD quantile approximation). *Let $\mu = 0, \sigma = 1, 0 < \alpha, \delta < \infty$ and $\lambda > 0$. Then the $(1 - 1/t)$ quantile of the symmetric GHD can be approximated by*

$$x_G^*=F_{Gamma}^{-1}\left(1-\frac{\delta^2 2K_\lambda(\alpha\delta)\alpha^\lambda}{t\Gamma(\lambda)}\right),\tag{4.8}$$

*whereby $F^{-1}_{Gamma}$ denotes the inverse function of the Incomplete Gamma Distribution with parameters $\lambda$ and $\frac{1}{\alpha}$.*

*Proof*: *Consider*

$$\begin{aligned} & \int_{x_G^*}^{\infty} h(y,\lambda,\alpha,\delta)dy = \frac{1}{t} \\ \Longleftrightarrow \quad & \int_{x_G^*}^{\infty} \frac{\alpha^{\lambda}}{\delta^{\lambda}\sqrt{2\pi}K_{\lambda}(\alpha\delta)} \frac{K_{\lambda-0.5}\left(\alpha\sqrt{\delta^2+y^2}\right)\left(\sqrt{\delta^2+y^2}\right)^{\lambda-0.5}}{\alpha^{\lambda-0.5}} dy = \frac{1}{t} \end{aligned}$$

*For large quantiles $x_G^*$ we use $\sqrt{\delta^2+y^2} \approx y$ and with (2.1) the above formula simplifies to*

$$\begin{aligned} & \int_{x_G^*}^{\infty} \frac{\sqrt{\alpha}}{\delta^{\lambda}\sqrt{2\pi}K_{\lambda}(\alpha\delta)} y^{\lambda-0.5} K_{\lambda-0.5}(\alpha y)dy \simeq \frac{1}{t} \\ \Longleftrightarrow \quad & \frac{\sqrt{\alpha}}{\delta^{\lambda}\sqrt{2\pi}K_{\lambda}(\alpha\delta)} \int_{x_G^*}^{\infty} y^{\lambda-0.5}\sqrt{\frac{\pi}{2\cdot y}} e^{-\alpha\cdot y} dy \simeq \frac{1}{t} \end{aligned}$$

*resp.*

$$\begin{aligned} & \frac{1}{\delta^{\lambda}2K_{\lambda}(\alpha\delta)} \int_{x_G^*}^{\infty} y^{\lambda-1}e^{-\alpha y}dy \simeq \frac{1}{t} \\ \Longleftrightarrow \quad & \frac{\Gamma(\lambda)}{\delta^{\lambda}2K_{\lambda}(\alpha\delta)\alpha^{\lambda}} \int_{x_G^*}^{\infty} \frac{1}{\left(\frac{1}{\alpha}\right)^{\lambda}\Gamma(\lambda)} y^{\lambda-1}e^{-\alpha y}dy \simeq \frac{1}{t}. \end{aligned}$$

*Using the cdf of the Gamma distribution ($F_{Gamma}$) we can derive the quantile approximation as follows:*

$$\begin{aligned} & \frac{\Gamma(\lambda)}{\delta^{\lambda}2K_{\lambda}(\alpha\delta)\alpha^{\lambda}} \left(1 - F_{Gamma}\left(x_G^*, a=\lambda, b=\frac{1}{\alpha}\right)\right) \simeq \frac{1}{t} \\ \Longleftrightarrow \quad & x_G^* \simeq F^{-1}_{Gamma}\left(1 - \frac{\delta^{\lambda}2K_{\lambda}(\alpha\delta)\alpha^{\lambda}}{t\Gamma(\lambda)}\right) \end{aligned}$$

*Because $\Gamma(\cdot)$ is just defined for values greater than zero we have to impose this restriction on $\lambda$. Moreover, because we approximate the Bessel function with (2.1), the quality of our formula decreases with increasing size of $\lambda$.*

□

To get some first expressions, we compare our result with an algorithm implemented in **R** by Wolfgang Breymann and David Lüthi from the Institute of Data Analyzes and Process Design at Züricher Hochschule für angewandte Wissenschaften which was the only available algorithm. As figure 4.3 and table 4.3 below reveal, our quantiles are slightly higher than that of R, at least for this parameter set.

## 5 Summary

The symmetric generalized hyperbolic distribution (GHD) as a flexible distribution model which is able to rebuild various kinds of tail behaviour. It also includes the Student t distribution and the Gaussian distribution as limiting cases. Within in this work we derived two separate tail quantile approximations for the GHD and the Student t which might be useful for fast and effective calculation of risk measure, for example. Comparisons with existing numerical integration routines are included as well.

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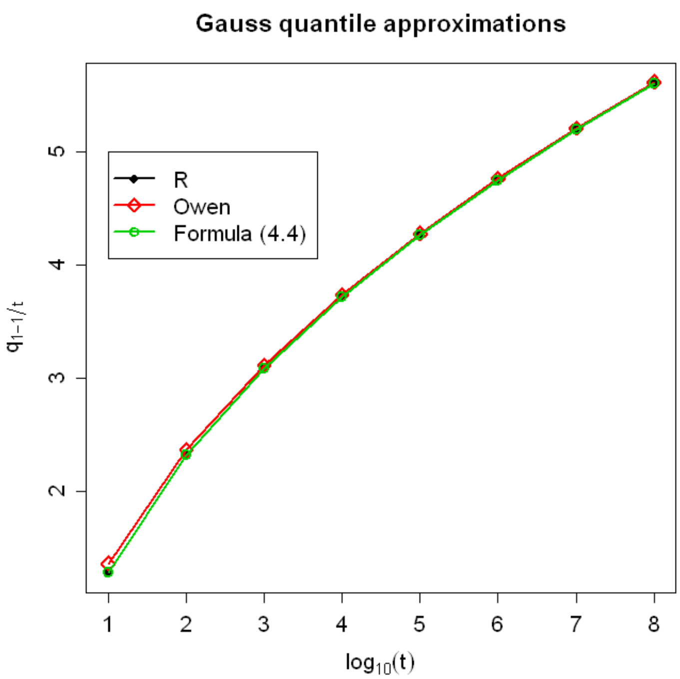
Table 1: The Gaussian quantile ($\mu = 0, \sigma = 1$)

| $\alpha$ | 0.9 | 0.99 | 0.999 | 0.9999 | 0.99999 | 0.999999 | 0.9999999 | 0.99999999 |
|---|---|---|---|---|---|---|---|---|
| $q_\alpha^R$ | 1.28155 | 2.32635 | 3.09023 | 3.71902 | 4.26489 | 4.75342 | 5.19934 | 5.61200 |
| $q_\alpha^{Owen}$ | 1.36192 | 2.36626 | 3.11647 | 3.73841 | 4.28019 | 4.76601 | 5.20999 | 5.62121 |
| $q_\alpha$ | 1.28155 | 2.32635 | 3.09023 | 3.71902 | 4.26489 | 4.75342 | 5.19934 | 5.61200 |

*The symbol $\alpha$ denotes the quantile point, $q_\alpha^R$ the values of R and $q_\alpha^{Owen}$ the values calculated by Owen (1962). Finally $q_\alpha$ corresponds to formula (4.4) .*

Figure 1: The Gaussian quantile



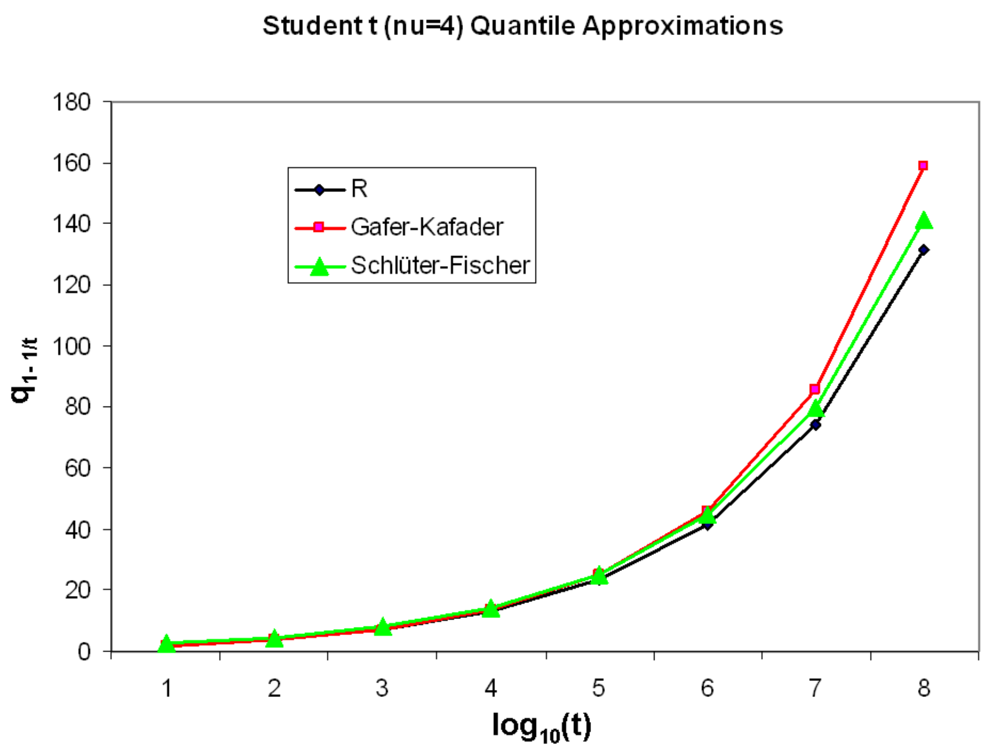

*On the ordinate, $q_{(1-1/t)}$ denotes the $(1 - 1/t)$ quantile. The parameters are $\mu = 0, \sigma = 1$.*

Table 2: The Student t quantile

| $\alpha$ | 0.9 | 0.99 | 0.999 | 0.9999 | 0.99999 | 0.999999 | 0.9999999 | 0.99999999 |
|---|---|---|---|---|---|---|---|---|
| $q_\alpha^R$ | 1.5332 | 3.7469 | 7.1732 | 13.0337 | 23.3322 | 41.5779 | 73.9858 | 131.5947 |
| $q_\alpha^{GK}$ | 1.5206 | 3.7398 | 7.2640 | 13.5081 | 24.9340 | 46.0813 | 85.4107 | 158.7470 |
| $q_\alpha$ | 2.5149 | 4.4721 | 7.9527 | 14.1421 | 25.1487 | 44.7214 | 79.5271 | 141.4214 |

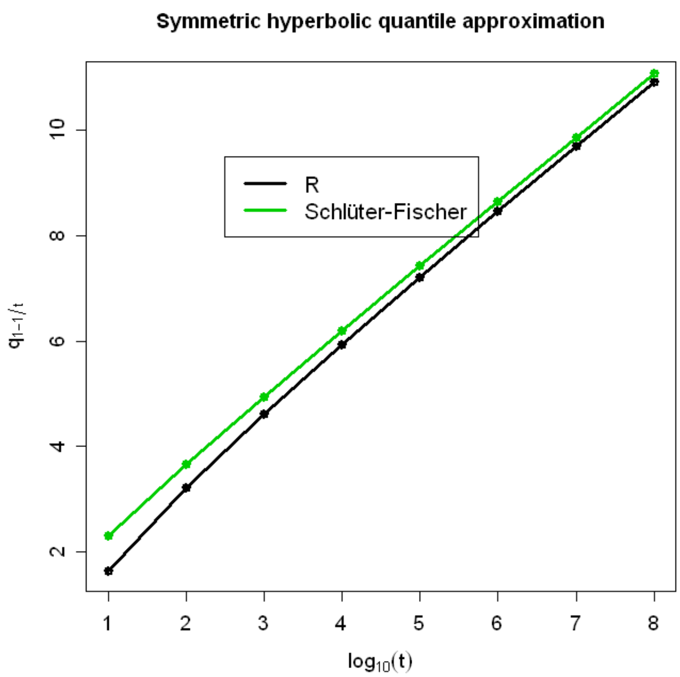
*The degrees of freedom,$\nu$, are 4. The symbol $\alpha$ denotes the quantile point, $q_\alpha^R$ the* **R** *approximation, $q_\alpha^{GK}$ the values of Gafer & Kafader (1984) and $q_\alpha$ our approximation.*

Figure 2: The Student t quantile ($\nu = 4$ degrees of freedom)




*On the ordinate, $q_{(1-1/t)}$ denotes the $(1-1/t)$ quantile.*

Figure 3: The GHD quantile




On the ordinate, $q_{(1-1/t)}$ denotes the $(1-1/t)$ quantile. The parameters are chosen as follows: $\alpha = 2$, $\delta = 2$, $\lambda = 2$.

Table 3: The GHD quantile

| $\alpha$ | 0.9 | 0.99 | 0.999 | 0.9999 | 0.99999 | 0.999999 | 0.9999999 | 0.99999999 |
|---|---|---|---|---|---|---|---|---|
| $q^1_\alpha$ | 1.6262 | 3.2167 | 4.6118 | 5.9305 | 7.2087 | 8.4619 | 9.6982 | 10.9223 |
| $q^2_\alpha$ | 2.3066 | 3.6539 | 4.9401 | 6.1953 | 7.4311 | 8.6542 | 9.8677 | 11.0740 |

The symbol $\alpha$ denotes the quantile point. $q^1_\alpha$ denotes the result of the algorithm implemented in R, $q^1_\alpha$ represents our result. The parameters used in this estimation are $\alpha = 2$, $\delta = 2$, $\lambda = 2$.

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