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Complete closed-form solution to a stochastic growth model and corresponding speed of economic recovery

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Abstract. We consider a continuous-time neoclassical one-sector stochastic growth model of Ramsey-type with CRRA utility and Cobb-Douglas technology, where *each* of the following components are exposed to exogeneous uncertainties (shocks): capital stock K, effectiveness of labor A, and labor force L; the corresponding dynamics is modelled by a system of three interrelated stochastic differential equations. For this framework, we solve *completely explicitly* the problem of a social planner who seeks to maximize expected lifetime utility of consumption. In particular, for *any* (e.g. short-term) time-horizon $t > 0$ we obtain in *closed form* the *sample paths* of the economy values K_t, A_t, L_t and the optimal consumption $c^{\mathrm{opt}}(K_t, A_t, L_t)$ as well as the non-equilibrium *sample paths* of the per capita effective capital stock $k_t = \frac{K_t}{A_t\, L_t}$. Moreover, we also deduce explicitly the limiting long-term behaviour of k_t expressed by the corresponding steady-state equilibrium distribution. As illustration, we present some Monte Carlo simulations where the abovementioned economy is considerably disturbed (out of equilibrium) by a sudden crash but recovers well within a realistic-size time-period.

Keywords: stochastic Ramsey-type growth; utility maximization; stochastic differential equations; explicit closed-form sample path dynamics; economic recovery; Monte Carlo simulations; steady-state.

1 Introduction

Nowadays, continuous-time neoclassical stochastic growth models of Ramsey type are well established tools for getting insights into the real mechanism through which uncertainties work their way through the economy, see e.g. the seminal work of Merton (1975) as well as the recent studies of Amilon and Bermin (2003), Roche (2003), Baten and Miah (2007), Smith (2007), Bucci et al. (2008), Posch (2009b); comprehensive coverages of this subject can be found e.g. in the books of Malliaris and Brock (1982), Turnovsky (2000b), Chang (2004), Wälde (2009a).

It is well known that the deduction of closed-form *pathwise* solutions to such continuous-time stochastic Ramsey growth models is not generally possible. Usually, for some components one can obtain closed-form *equilibrium* solutions – in the sense of explicitly derived steady-state distributions. Furthermore, one typically approximates in an *indirect* way the unknown solution sample paths by employing a corresponding numerical Euler method for the underlying stochastic differential equations (SDEs). However, in principle there can be pitfalls (e.g. non-convergence problems in case of wrong stepsize choices) to such an Euler-type approach even in the case of SDEs with very smooth drift and diffusion term, like for the prominent Cox-Ingersoll-Ross interest rate model (cf. Higham and Mao (2005)). Such potential difficulties can be quite reduced in case of finding an explicit pathwise solution to be sampled *directly* from.

Another advantage of finding explicit pathwise solutions to stochastic Ramsey growth models is that one gets the precise dynamics of involved economy components even if they *start* respectively *restart* from a *non-equilibrium* (i.e. non-steady-state) distribution. The latter situation certainly appears in case of a sudden considerable (partial) economy crash, whereas the former seems to be reasonable for modelling emerging economies like the BRIC countries and other, newly industrializing, countries. Within such a context, the following questions arise: how long does it take the model economy to recover back to – respectively to reach (e.g. for the first time) – a steady-state equilibrium distribution ? Is this recovery time of realistic length ?

Within a neoclassical one-sector economy framework (posed in Section 2) which consists of a continuous-time, "fully" stochastic, Ramsey-type growth model with CRRA utility and Cobb-Douglas technology, the main goals of this paper are

- to solve *completely explicitly* the social planner's problem – namely to maximize the expected lifetime utility of consumption under appropriate SDE-type constraints on capital stock K, labor-effectiveness A and labor force L – by obtaining in a closed form
  - the optimal consumption strategy (see Section 3),
  - the sample-path solutions K_t, A_t, L_t at time $t \geq 0$ of the underlying "optimized" stochastic differential equations, and hence the *non-equilibrium* sample paths of the per capita effective capital stock described by $k_t = \frac{K_t}{A_t L_t}$ (see Section 3),
  - the corresponding long-term limit respectively the *equilibrium* steady-state distribution of k_t (see Section 4),
- to study the corresponding speed of recovery from an economy crash by means of "direct" (i.e. non-Euler-method-type) Monte Carlo simulations for realistic parameter values (see Section 5).

2 The Economic Framework

In terms of the – at time $t \geq 0$ prevailing – capital stock K_t, effectiveness of labor A_t and labor force L_t, the production function for the economy is supposed to be of the Cobb-Douglas-type

$$Y_t = K_t^{\alpha}(A_t L_t)^{1-\alpha} \tag{1}$$

for some arbitrary but fixed capital share (output elasticity of capital stock) $\alpha \in (0,1)$. The random dynamics of the effectiveness of labor (labor-augmenting technology level, total factor productivity) A_t evolves according to a geometric Brownian motion given by the stochastic differential equation (SDE)

$$dA_t = \mu_A A_t dt + \sigma_A A_t dB_t^A, \qquad A_0 > 0 \text{ given}, \tag{2}$$

with average growth rate $\mu_A \geq 0$ and constant volatility $\sigma_A > 0$. Furthermore, the random dynamics of the labor force (labor supply) L_t is represented by another geometric Brownian motion solving the SDE

$$dL_t = \mu_L L_t dt + \sigma_L L_t dB_t^L, \qquad L_0 > 0 \text{ given}, \tag{3}$$

with average growth rate $\mu_L \in \mathbb{R}$ and constant volatility $\sigma_L > 0$. Finally, the random dynamics of the capital stock K_t evolves according to

$$dK_t = \big[Y_t - \delta K_t - L_t\, c_t\big]dt + \sigma_K K_t dB_t^K, \qquad K_0 > 0 \text{ given}, \tag{4}$$

where $\sigma_K > 0$ denotes the constant volatility, $\delta \geq 0$ the constant depreciation rate and c_t the random per capita consumption at time t.

Notice that we model our economy to be "fully stochastic" in the sense that (due to the assumption $\sigma_A > 0$, $\sigma_L > 0$, $\sigma_K > 0$) *each* of the economy values A_t, L_t, K_t is exposed to a source of uncertainties/shocks, here in form of *independent* standard Brownian motions B_t^A, B_t^L, B_t^K. By employing (1), the dynamics (4) turns into the SDE[1]

$$dK_t = \big[K_t^{\alpha}(A_t L_t)^{1-\alpha} - \delta K_t - L_t\, c_t\big]dt + \sigma_K K_t dB_t^K, \qquad K_0 > 0 \text{ given}. \tag{5}$$

The representative consumer is supposed to have a constant rate $\rho \geq 0$ of time-preference and CRRA utility[2]

$$u(c) = \frac{c^{1-\gamma} - 1}{1-\gamma} \tag{6}$$

with coefficient of relative risk aversion (respectively, the reciprocal of the intertemporal elasticity of substitution) γ. For the rest of this paper, we assume that the coefficient of relative risk aversion is equal to the capital share, i.e. $\gamma = \alpha$. This equality is well established in

[1] for technical reasons, we set (hiddenly) in advance both the drift term and the diffusion term to be zero in case that K_t turns strictly negative; later on it will however turn out that the solution satisfies $K_t > 0$ with probability 1 which is what one expects from capital stock

[2] we have chosen a power type form which converges to the logarithmic function as γ tends to zero

literature, see e.g. Chang (1988), Xie (1991, 1994), Boucekkine and Ruiz-Tamarit (2004), Wälde (2005), Smith (2006, 2007), Boucekkine and Ruiz-Tamarit (2008), Ferrara and Guerrini (2009a,b), Posch (2009a,b), Posch and Wälde (2010).

In order to get precise and short mathematical formulations of the involved optimization problem, we introduce the notation $X_t = (K_t, A_t, L_t)$ for the time-t economy values. Furthermore, we use P to denote the underlying probability law [3] governing the economy-value process $(X_t)_{t\geq 0}$, and by E the corresponding expectation. The term $\mathrm{P}^{s,x}$ stands for the conditional probability law given that the time-s economy value X_s is x, with deterministic three-dimensional vector x; in other words, $\mathrm{P}^{s,x}[\,\cdot\,] = \mathrm{P}[\,\cdot\,|\,X_s = x\,]$. For the corresponding conditional expectation we write $\mathrm{E}^{s,x}$, i.e. $\mathrm{E}^{s,x}[\,\cdot\,] = \mathrm{E}[\,\cdot\,|\,X_s = x\,]$.

Within the abovementioned economy framework, the social planner is supposed to solve the following optimization problem for fixed $\gamma \in (0,1)$, depreciation rate $\delta \geq 0$, time-preference rate $\rho \geq 0$, volatilities $\sigma_K, \sigma_A, \sigma_L > 0$, and average growth rates $\mu_A \geq 0$, $\mu_L \in \mathbb{R}$, under the technical assumption

$$(1-\gamma)\mu_A + \mu_L < \rho \tag{7}$$

which we require to hold for the rest of this paper.

Optimization Problem 2.1. *For prevailing economy value $x > 0$ [4] at arbitrary decision time $s \geq 0$, maximize over all admissible nonnegative per capita consumption strategies $(c_t)_{t\geq s} \in \mathcal{C}(s,x)$ [5] the expected (discounted) remaining-lifetime utility of consumption*

$$\mathrm{E}^{s,x}\Big[\int_s^{\tau_s} \mathrm{e}^{-\rho t}\,\frac{c_t^{1-\gamma}-1}{1-\gamma}\,L_t\,dt\Big] \tag{8}$$

subject to the constraints (5), (2), (3).

Here, τ_s denotes is the first time $r > s$ for which K_r becomes nonpositive. The corresponding value function is denoted by $V(s,x)$, i.e.

[3] on the sample space Ω of all continuous functions $\omega : [0,\infty) \mapsto \mathbb{R}\times[0,\infty)\times[0,\infty)$ from the time-interval $[0,\infty)$ into the three-dimensional Euclidean subspace $\mathbb{R}\times[0,\infty)\times[0,\infty)$; each such sample point $\omega \in \Omega$ is identified with a particular potential economy-value-(path-)scenario from time 0 to time ∞. As usual we omit ω except for Lemma A.1 in Appendix A.2 as well as its later point of application.

[4] here and henceforth, $x > 0$ means here that all three components of x are strictly positive; the same terminology will also be used for vectors other than x.

[5] as usual, for technical reasons, we confine ourselves here to a "sufficiently (maximally) large" collection $\mathcal{C}(s,x)$ of so called *admissible* consumption strategies $(c_t)_{t\geq s}$ which are nonnegative (adapted cadlag) processes such that (i) all the involved quantities are well defined as well as finite wherever needed, (ii) the system (5), (2), (3) of SDEs has a unique strong solution, (iii) the optimization problem can be carried out and verified afterwards. Notice that from the well-known distributional properties of the geometric Brownian motion L_t (cf. (3)) it is easy to see that the value function $V(s,x)$ is strictly larger than $-\infty$ for all (s,x).

$$V(s,x) = \sup_{(c_t)_{t\geq s}\in\mathcal{C}(s,x)} \mathrm{E}^{s,x}\Big[\int_s^{\tau_s} \mathrm{e}^{-\rho t}\,\frac{c_t^{1-\gamma}-1}{1-\gamma}\,L_t\,dt\Big]\,. \tag{9}$$

In the following, without (much) loss of generalization we always suppose that c_t depends on $(K_u)_{u\geq 0}$, $(A_u)_{u\geq 0}$, $(L_u)_{u\geq 0}$ only in a time-homogeneous "memoryless" ("Markovian") way, i.e. $c_t = c(K_t, A_t, L_t)$ for some deterministic, nonnegative consumption strategy $c(\cdot) \in \mathcal{C}$ [6]. Accordingly, with the help of the abovementioned assumption $\alpha = \gamma$, the SDE (5) turns into

$$dK_t = \big[K_t^{\gamma}(A_tL_t)^{1-\gamma} - \delta K_t - L_t\,c(X_t)\big]dt + \sigma_K K_t dB_t^K\,, \qquad K_0 > 0 \text{ given}\,, \tag{10}$$

and one can simplify

$$V(s,x) = \mathrm{e}^{-\rho s}\,V(0,x) = \mathrm{e}^{-\rho s}\sup_{c(\cdot)\in\mathcal{C}} \mathrm{E}^{0,x}\Big[\int_0^{\tau_0} \mathrm{e}^{-\rho t}\,\frac{\big(c(X_t)\big)^{1-\gamma}-1}{1-\gamma}\,L_t\,dt\Big]\,. \tag{11}$$

Hence, for the rest of this paper, we confine ourselves to this special case of Optimization Problem 2.1:

find the value function $V(0,x)$ (inclusively the optimizing consumption strategy $c^{opt}(\cdot)$) subject to the constraints (10), (2), (3). (12)

In Table 1 we exemplarily mention some studies about continuous-time stochastic growth models (mainly with Cobb-Douglas type technology), which are similiar but of lower-dimensional "uncertainty" than our framework; for comparison, we have also included some Solow-Swan type investigations.

reference	σ_K	σ_A	μ_A	σ_L	μ_L	type
Amilon and Bermin (2003)	$=0$			>0	$\in\mathbb{R}$	Ramsey
Baten and Miah (2007)	$=0$			>0	$\in\mathbb{R}$	Ramsey
Bourguignon (1974)	>0			>0	$\in\mathbb{R}$	Solow–Swan
Bucci et al. (2008)	$=0$	>0	$\in\mathbb{R}$	$=0$	$\in\mathbb{R}$	Ramsey
Jensen and Richter (2007)	>0			>0	$\in\mathbb{R}$	Solow–Swan
Merton (1975)	$=0$			>0	$\in\mathbb{R}$	Solow–Swan & Ramsey
Posch (2009b)	$=0$	>0	$\in\mathbb{R}$			Ramsey
Roche (2003)	>0			>0	$\in\mathbb{R}$	Ramsey
Smith (2007)	$=0$	>0	$\in\mathbb{R}$			Ramsey
Smith (2007)	$=0$			>0	$\in\mathbb{R}$	Ramsey
this paper	>0	>0	$\in\mathbb{R}$	>0	$\in\mathbb{R}$	Ramsey

Table 1: Some literature comparison: "blank" spaces mean no-occurrence; eventual parameter restrictions are not mentioned here.

Some further related investigations, which are less similar to ours than those mentioned in Table 1, can be found in e.g. Eaton (1981), Chang (1988), Pindyck and Solimano (1993), Corsetti (1997), Grinols and Turnovsky (1998), Attanasio (1999), Jensen and Wang (1999), Pindyck (2000), Turnovsky (2000a), Bank and Riedel (2001), Jensen et al. (2001), Cagetti

[6] such that $(c_t)_{t\geq s} \in \mathcal{C}(s,x)$ holds

et al. (2002), Fleming and Stein (2004), Kenc (2004), Maenhout (2004), Clemens (2005), Fleming and Pang (2005), Hartley and Rogers (2005), Jensen and Larsen (2005), Jones and Manuelli (2005), Steger (2005), Wälde (2005), Sennewald and Wälde (2006), Turnovsky and Smith (2006), Nishide and Ohyama (2007), Stein (2007), Elie and Touzi (2008), Morimoto (2008), Morimoto and Zhou (2008), Dunbar (2009), Posch (2009a), Sotomayor and Cadenillas (2009), Wälde (2009b), Chen and Wu (2010), Palacios (2010) and Posch and Wälde (2010).

3 Closed-Form Optimal Consumption and Closed-Form Economy Sample Paths

By virtue of (11), it suffices to solve the Optimization Problem 2.1 for the special case $s = 0$ (cf. (12)), for which we can identify x with the initial economy values, i.e. $x = (K_0, A_0, L_0) > 0$. The corresponding unique optimal consumption strategy turns out to be $c_t^{\text{opt}} = c^{\text{opt}}(X_t)$ where $c^{\text{opt}}(\cdot)$ is given in terms of $X = (K, A, L) > 0$ [7] by

$$c^{\text{opt}}(X) = \tfrac{1}{\gamma}\left[\rho + \delta(1-\gamma) - \gamma\,\mu_L + \tfrac{1}{2}(1-\gamma)\,\gamma\,(\sigma_K^2 + \sigma_L^2)\right]\frac{K}{L} =: b_1\frac{K}{L}\ ; \tag{13}$$

this will be derived in Appendix A.1 by solving the corresponding Hamilton-Jacobi-Bellman equation explicitly, and verified in Appendix A.4 thereafter. Notice that

- the optimal consumption strategy $c^{\text{opt}}(\cdot)$ does not depend on the labor-effectiveness (technology level) A,
- from the *general* parameter restrictions $\gamma \in (0,1)$, $\delta \geq 0$, $\sigma_K > 0$, $\sigma_L > 0$ as well as assumption (7) one gets $b_1 > 0$ and thus $c^{\text{opt}}(X) > 0$.

By formally plugging $c_t^{\text{opt}} = c^{\text{opt}}(X_t)$ into (10), one arrives at

$$dK_t = \left[K_t^{\gamma}(A_tL_t)^{1-\gamma} - (\delta + b_1)K_t\right]dt + \sigma_K K_t dB_t^K\ , \qquad K_0 > 0 \text{ given.} \tag{14}$$

It is possible to obtain *explicitly* the closed form of the unique strong solution $X_t = (K_t, A_t, L_t) > 0$ of the coupled system (14), (2), (3) of SDEs, and thus the desired non-equilibrium sample-path dynamics of the economy values. In order to achieve this, let us first notice that (2), (3) are uniquely solved by the well-known geometric Brownian motions

$$A_t = A_0\,\mathrm{e}^{(\mu_A - \frac{1}{2}\sigma_A^2)t + \sigma_A B_t^A} > 0 \qquad \text{and} \tag{15}$$

$$L_t = L_0\,\mathrm{e}^{(\mu_L - \frac{1}{2}\sigma_L^2)t + \sigma_L B_t^L} > 0\,. \tag{16}$$

Furthermore, as a next step we seek for the unique strong solution k_t of the SDE

$$dk_t = \left[k_t{}^{\gamma} - b_2k_t\right]dt + k_t\left[\sigma_K dB_t^K - \sigma_A dB_t^A - \sigma_L dB_t^L\right]\ , \qquad k_0 > 0 \text{ given,} \tag{17}$$

[7] with slight abuse of notation

where

$$b_2 := \delta + b_1 + \mu_A + \mu_L - \sigma_A^2 - \sigma_L^2 = \tfrac{\rho+\delta}{\gamma} + \mu_A + \tfrac{1}{2}(1-\gamma)\sigma_K^2 - \sigma_A^2 - \tfrac{1}{2}(1+\gamma)\sigma_L^2 \qquad (18)$$

can be negative, zero or positive, depending on the parameter constellations. Notice that we start here with a single-valued distribution – namely with mass concentrated at $k_0 > 0$ – rather than the equilibrium steady-state distribution[8]. In other words, the process $(k_t)_{t\geq 0}$ starts in non-equilibrium which explains respectively justifies our correspondingly chosen terminology throughout this paper. As it will be derived in Appendix A.2, the unique strong (non-equilibrium) solution of (17) is given by

$$\begin{aligned} k_t &= \mathrm{e}^{\sigma_K B_t^K - \sigma_A B_t^A - \sigma_L B_t^L - \frac{1}{2} b_3 t} \\ &\quad \cdot \left[(1-\gamma) \int_0^t \mathrm{e}^{(1-\gamma)\left[-\sigma_K B_s^K + \sigma_A B_s^A + \sigma_L B_s^L + \frac{1}{2} b_3 s\right]} \, ds + {k_0}^{1-\gamma} \right]^{\frac{1}{1-\gamma}}, \qquad (19) \end{aligned}$$

$$\text{with } b_3 := \sigma_K^2 + \sigma_A^2 + \sigma_L^2 + 2b_2 = 2\tfrac{\rho+\delta}{\gamma} + 2\mu_A + (2-\gamma)\sigma_K^2 - \sigma_A^2 - \gamma\sigma_L^2 . \qquad (20)$$

Because of $\gamma \in (0,1)$ and $k_0 > 0$, one gets with probability 1 that $k_t > 0$ for all $t \geq 0$. Notice that b_3 can be negative, zero or positive, depending on the parameter constellations. From (19), (15), (16) we construct the process $(K_t)_{t\geq 0}$ by $K_t := k_t\, A_t\, L_t$. Accordingly, by means of the appropriate three-dimensional version of Ito's formula one can derive from (17), (2), (3) that $(K_t)_{t\geq 0}$ is the unique strong solution of the SDE (14); see Appendix A.3 for the details. Since K_t represents the capital stock at time t, one can interpret $k_t = \frac{K_t}{A_t L_t}$ as *per capita effective capital stock* (capital per effective worker, capital per efficiency unit) at time t. [9]

Summing up, with the help of (19) we arrive at the *explicit closed form* of the *sample paths* of the economy value process given by $X_t = (K_t, A_t, L_t)$ with (15), (16) and

$$\begin{aligned} K_t &= k_t\, A_t\, L_t \qquad (21) \\ &= A_0 L_0\, \mathrm{e}^{\left[\mu_L - \frac{\rho+\delta}{\gamma} - \frac{1}{2}(2-\gamma)\sigma_K^2 - \frac{1}{2}(1-\gamma)\sigma_L^2\right] t + \sigma_K B_t^K} \\ &\quad \cdot \left[(1-\gamma) \int_0^t \mathrm{e}^{(1-\gamma)\left[-\sigma_K B_s^K + \sigma_A B_s^A + \sigma_L B_s^L + \frac{1}{2} b_3 s\right]} \, ds + \left(\tfrac{K_0}{A_0 L_0}\right)^{1-\gamma} \right]^{\frac{1}{1-\gamma}}. \qquad (22) \end{aligned}$$

One gets immediately with probability 1 that $K_t > 0$ for all $t \geq 0$, and hence the random remaining lifetime in Optimization Problem 2.1 with $s = 0$ (cf. (12)) becomes $\tau_0 = \infty$. In a similar way, one can deduce from (19) and (15) a closed-form sample-path expression for the optimal consumption $c_t^{\mathrm{opt}} = c^{\mathrm{opt}}(X_t) = b_1\, k_t\, A_t$ (cf. (13), (21)).

Let us further study the abovementioned economy value processes. Of course – due to their geometric Brownian motion nature (15) respectively (16) – the behaviour of the labor-

[8] which will be derived later on

[9] vice versa, by using the appropriate version of Ito's formula for $k_t = \frac{K_t}{A_t L_t}$ one can deduce from (14), (2), (3) that k_t is a solution of the SDE (17); since this is purely formal until one knows the existence of a solution K_t of (14), we have used above the opposite direction of arguments.

effectiveness process $(A_t)_{t\geq 0}$ respectively the labor-force process $(L_t)_{t\geq 0}$ is very well known in detail; for instance, A_t respectively L_t is lognormal distributed for each time $t \geq 0$. Thus, in the following we concentrate on deriving some properties of the capital stock process $(K_t)_{t\geq 0}$ and its per capita effective counterpart $(k_t)_{t\geq 0}$. We first notice that by means of the explicit representation (22) one can straightforwardly perform corresponding Monte Carlo simulations upon future capital stock evolution scenarios; this will be done in Section 5 below.

Next, we investigate the corresponding probability distribution of the non-equilibrium per capita effective capital stock k_t for any fixed (e.g. short-term or long-term) horizon $t \geq 0$; the analogous conclusions for the capital stock K_t can be then deduced in principle from (21), (15), (16).

It is straightforward to deduce from (19) that the distribution of k_t is given by

$$k_t \sim \Big[Y_{1,t}\Big\{\theta_1 Y_{2,t} + \theta_2\Big\}\Big]^{\frac{1}{1-\gamma}} \tag{23}$$

with constants $\theta_1 := 1-\gamma > 0$, $\theta_2 := {k_0}^{1-\gamma} > 0$ as well as random variables $Y_{1,t}$, $Y_{2,t}$ with the following properties in terms of the newly introduced variables $\breve{\sigma} := (1-\gamma)\sqrt{\sigma_K^2 + \sigma_A^2 + \sigma_L^2}$ and $b_4 := \frac{b_3}{2\,(1-\gamma)(\sigma_K^2+\sigma_A^2+\sigma_L^2)}$:

- $Y_{1,t} \sim \mathcal{LN}(-\breve{\sigma}^2 b_4 t, \breve{\sigma}^2 t)$, i.e. $\log Y_{1,t}$ is normal distributed with mean $-\breve{\sigma}^2 b_4 t$ and variance $\breve{\sigma}^2 t$,
- $Y_{2,t} \sim \int\limits_0^t \mathrm{e}^{\breve{\sigma}^2 b_4 s - (1-\gamma)\left[\sigma_K B_s^K - \sigma_A B_s^A - \sigma_L B_s^L\right]}\, ds$, which in case of the assumption $b_4 > -1$ respectively its equivalent characterizations

  $$b_3 > -2\,(1-\gamma)(\sigma_K^2 + \sigma_A^2 + \sigma_L^2) \iff 2\tfrac{\rho+\delta}{\gamma} + 2\mu_A + (4-3\gamma)\sigma_K^2 + (1-2\gamma)\sigma_A^2 + (2-3\gamma)\sigma_L^2 > 0 \tag{24}$$

  has the following [10] density $f(\cdot)$, cf. Borodin and Salminen (2002, p. 612, formula 1.8.4):

$$f(y) := \frac{\breve{\sigma}^{2b_4+1} y^{b_4-\frac{1}{2}}}{2^{b_4+\frac{1}{2}} {k_0}^{b_4+\frac{1}{2}}}\; M_{\frac{1}{2}\breve{\sigma}^2 t}\big(b_4 - \tfrac{1}{2}, \tfrac{k_0}{\breve{\sigma}^2 y}\big)\; \exp\big\{-\tfrac{1}{2} b_4^2 \breve{\sigma}^2 t - \tfrac{k_0}{\breve{\sigma}^2 y}\big\}, \quad y \geq 0\,.$$

Here, one involves the *transformed Kummer function* which for $\mu > -\frac{3}{2}$ and $z > 0$ is defined by

$$M_y(\mu, z) := \frac{8 z^{\frac{3}{2}} \Gamma(\mu + \frac{3}{2})\, \mathrm{e}^{\frac{\pi^2}{4y}}}{\pi\sqrt{2\pi y}} \int\limits_0^\infty \mathrm{e}^{-z\,\mathrm{ch}(2u) - \frac{u^2}{y}}\, \widetilde{M}\big(-\mu, \tfrac{3}{2}, 2z\,\mathrm{sh}^2 u\big)\, \mathrm{sh}(2u) \sin\big(\tfrac{\pi u}{y}\big) du\,,$$

with Kummer function $\widetilde{M}(a,b,x) := 1 + \sum\limits_{k=1}^\infty \frac{a(a+1)\ldots(a+k-1)x^k}{b(b+1)\ldots(b+k-1)k!}$ as well as the hyperbolic functions $\mathrm{ch}\, x = \frac{1}{2}(\mathrm{e}^x + \mathrm{e}^{-x}) = \sum\limits_{k=0}^\infty \frac{x^{2k}}{2k!}$ and $\mathrm{sh}\, x = \frac{1}{2}(\mathrm{e}^x - \mathrm{e}^{-x}) = \sum\limits_{k=0}^\infty \frac{x^{2k+1}}{(2k+1)!}$. Moreover, we obtain the moments

[10] Lebesgue

$$
\begin{aligned}
\mathrm{E}\big[{k_t}^{1-\gamma}\big] &= E\Big[(1-\gamma)\int_0^t \mathrm{e}^{(1-\gamma)\left[\sigma_K(B_t^K-B_s^K)-\sigma_A(B_t^A-B_s^A)-\sigma_L(B_t^L-B_s^L)-\frac{1}{2}b_3(t-s)\right]}\,ds\Big] \\
&\quad + \mathrm{E}\Big[\mathrm{e}^{(1-\gamma)\left[\sigma_K B_t^K-\sigma_A B_t^A-\sigma_L B_t^L-\frac{1}{2}b_3 t\right]}\,{k_0}^{1-\gamma}\Big] \\
&= (1-\gamma)\int_0^t \mathrm{E}\Big[\mathrm{e}^{(1-\gamma)\left[\sigma_K(B_t^K-B_s^K)-\sigma_A(B_t^A-B_s^A)-\sigma_L(B_t^L-B_s^L)-\frac{1}{2}b_3(t-s)\right]}\Big]ds \\
&\quad + {k_0}^{1-\gamma}\,\mathrm{E}\Big[\mathrm{e}^{(1-\gamma)\left[\sigma_K B_t^K-\sigma_A B_t^A-\sigma_L B_t^L-\frac{1}{2}b_3 t\right]}\Big] \\
&= (1-\gamma)\int_0^t \mathrm{e}^{\left[-\frac{1}{2}(1-\gamma)b_3+\frac{1}{2}(1-\gamma)^2(\sigma_K^2+\sigma_A^2+\sigma_L^2)\right](t-s)}\,ds \\
&\quad + {k_0}^{1-\gamma}\,\mathrm{e}^{\left[-\frac{1}{2}(1-\gamma)b_3+\frac{1}{2}(1-\gamma)^2(\sigma_K^2+\sigma_A^2+\sigma_L^2)\right]t} \\
&= (1-\gamma)\int_0^t \mathrm{e}^{b_5(t-s)}\,ds + {k_0}^{1-\gamma}\,\mathrm{e}^{b_5 t} \quad =: \varkappa_t > 0 \qquad (25)
\end{aligned}
$$

with

$$
\begin{aligned}
b_5 &:= -\tfrac{1}{2}(1-\gamma)b_3 + \tfrac{1}{2}(1-\gamma)^2(\sigma_K^2+\sigma_A^2+\sigma_L^2) \\
&= \tfrac{1}{2}(1-\gamma)\Big\{-2\tfrac{\rho+\delta}{\gamma} - 2\mu_A - \sigma_K^2 + (2-\gamma)\sigma_A^2 + \sigma_L^2\Big\}. \qquad (26)
\end{aligned}
$$

In the case $b_5 = 0$ one gets

$$
\varkappa_t = (1-\gamma)t + {k_0}^{1-\gamma} \qquad (27)
$$

and for $b_5 \neq 0$

$$
\varkappa_t = -\tfrac{1-\gamma}{b_5} + \mathrm{e}^{b_5 t}\cdot\Big(\tfrac{1-\gamma}{b_5} + {k_0}^{1-\gamma}\Big)\ . \qquad (28)
$$

From this, one can deduce a lower bound for the expectation of the per capita effective capital stock via Jensen's inequality

$$
\mathrm{E}\big[k_t\big] \geq \big\{\mathrm{E}\big[{k_t}^{1-\gamma}\big]\big\}^{\frac{1}{1-\gamma}} = \varkappa_t^{\frac{1}{1-\gamma}}, \quad t \geq 0. \qquad (29)
$$

4 Long-Term Limit and Steady State of the Stochastic Non-Equilibrium Capital Stock

Recall that we know *explicitly* the closed form of the capital stock process $(K_t)_{t\geq 0}$ (cf. (22)), as well as of the non-equilibrium per capita effective capital stock process $(k_t)_{t\geq 0}$ (cf. (19)). For the latter, we now present the corresponding time-asymptotics and stationarity properties. Those will be deduced in Appendix A.5 below in the usual *implicit* way by means of scaling and speed measure techniques. Accordingly, starting from the SDE

$$
dk_t = \big[{k_t}^{\gamma} - b_2 k_t\big]dt + k_t\,\big[\sigma_K dB_t^K - \sigma_A dB_t^A - \sigma_L dB_t^L\big] \qquad \text{(cf. (17))}
$$

we obtain under the (here, equivalently characterized) assumption

$$b_3 > 0 \iff 2\tfrac{\rho+\delta}{\gamma} + 2\mu_A + (2-\gamma)\sigma_K^2 - \sigma_A^2 - \gamma\sigma_L^2 > 0 \tag{30}$$

– which is stronger than assumption (24) – for every initial value $k_0 \in (0,\infty)$ the (infinite-time-horizon-)*limit* distribution function

$$F_{lim}(z) := \lim_{t\to\infty} \mathrm{P}^{0,k_0}\big[k_t \le z\big] = (1-\gamma)\frac{{b_6}^{b_7}}{\Gamma(b_7)} \int_0^z \exp\big\{-b_6 k^{\gamma-1}\big\} k^{-[b_7(1-\gamma)+1]} dk\,, \qquad z \in (0,\infty), \tag{31}$$

with

$$b_6 := \tfrac{2}{(\sigma_K^2+\sigma_A^2+\sigma_L^2)\cdot(1-\gamma)} > 0 \qquad \text{and} \tag{32}$$

$$b_7 := \frac{2\frac{\rho+\delta}{\gamma} + 2\mu_A + (2-\gamma)\sigma_K^2 - \sigma_A^2 - \gamma\sigma_L^2}{(\sigma_K^2+\sigma_A^2+\sigma_L^2)\cdot(1-\gamma)} > 0\,. \tag{33}$$

Notice that the limit distribution function $F_{lim}(\cdot)$ of $(k_t)_{t\ge 0}$ does not depend on the initial value k_0. The corresponding distribution (i.e. probability law) will be denoted by P_{lim}; according to (31) it has the density

$$f_{lim}(k) = (1-\gamma)\frac{{b_6}^{b_7}}{\Gamma(b_7)} \exp\big\{-b_6 k^{\gamma-1}\big\} k^{-[b_7(1-\gamma)+1]}\,, \qquad k \in (0,\infty)\,. \tag{34}$$

In contrast, one can also show that the per capita effective capital stock process $(k_t)_{t\ge 0}$ has a steady-state distribution P_{stea} – also known as stationary distribution, invariant distribution, equilibrium distribution – which means in particular that if at some time $u \ge 0$ the distribution of k_u is equal to P_{stea}, then at any later time $v > u$ the distribution of k_v is also equal to P_{stea}. Indeed, one gets $P_{stea} = P_{lim}$, i.e. the steady-state distribution and the limit distribution coincide. Hence, the steady-state distribution P_{stea} has distribution function $F_{stea}(z) = F_{lim}(z)$ given in (31) and density $f_{stea}(k) = f_{lim}(k)$ given in (34). In the light of this, within our framework there holds the effect that for long-term (but finite) time horizons u the distribution of k_u becomes close to P_{stea}, and thus at any later time horizon $v > u$ the distribution of k_v stays close to P_{stea}. Despite of this (approximative) *equilibrium* effect, we emphasize again that in our context the per capita effective capital stock process $(k_t)_{t\ge 0}$ *starts* in non-equilibrium – namely, with the single-valued distribution at k_0 – rather than with the steady-state distribution P_{stea}.

Let us mention here that the above equilibrium investigations extend in a certain sense the similar continuous-time steady-state studies under lower-dimensional uncertainty (cf. Table 1) carried out e.g. by Merton (1975), Smith (2007) within a Ramsey type setup, and e.g. by Bourguignon (1974), Merton (1975), Jensen and Richter (2007) within a Solow-Swan type setup. Also, these papers do not deal with finding explicit closed-form sample-path representations of the capital stock processes $(k_t)_{t\ge 0}$ and $(K_t)_{t\ge 0}$, which we have achieved in Section 3 above.

Next, we compute the mean μ_{stea} of the steady-state distribution P_{stea} (limit distribution P_{lim}) by

$$\mu_{stea} \;=\; (1-\gamma)\frac{{b_6}^{b_7}}{\Gamma(b_7)}\int\limits_0^\infty k\,\exp\{-b_6k^{\gamma-1}\}\,k^{-[b_7(1-\gamma)+1]}dk$$

$$= \frac{{b_6}^{b_7}}{\Gamma(b_7)}\int\limits_0^\infty \mathrm{e}^{-b_6y}\,y^{b_7-\frac{1}{1-\gamma}-1}dy \tag{35}$$

which is finite if and only if $b_7 > \frac{1}{1-\gamma}$. The latter is equivalent to

$$b_3 > \sigma_K^2 + \sigma_A^2 + \sigma_L^2 \tag{36}$$

which itself is equivalent to the original-parameters-involving condition

$$2\tfrac{\rho+\delta}{\gamma} + 2\mu_A > -(1-\gamma)\sigma_K^2 + 2\sigma_A^2 + (1+\gamma)\sigma_L^2\,. \tag{37}$$

Hence, under the requirement (36) respectively (37) – which is stronger than (30) and (24) – we end up with

$$\begin{aligned}
\mu_{stea} \;&=\; b_6{}^{\frac{1}{1-\gamma}}\,\frac{\Gamma(b_7-\frac{1}{1-\gamma})}{\Gamma(b_7)}\int\limits_0^\infty \frac{{b_6}^{b_7-\frac{1}{1-\gamma}}}{\Gamma(b_7-\frac{1}{1-\gamma})}\,\mathrm{e}^{-b_6y}\,y^{b_7-\frac{1}{1-\gamma}-1}dy \\
&=\; b_6{}^{\frac{1}{1-\gamma}}\,\frac{\Gamma(b_7-\frac{1}{1-\gamma})}{\Gamma(b_7)} \\
&=\; \Big(\tfrac{2}{(\sigma_K^2+\sigma_A^2+\sigma_L^2)(1-\gamma)}\Big)^{\frac{1}{1-\gamma}}\,\frac{\Gamma\Big(\frac{2\frac{\rho+\delta}{\gamma}+2\mu_A+(1-\gamma)\sigma_K^2-2\sigma_A^2-(1+\gamma)\sigma_L^2}{(\sigma_K^2+\sigma_A^2+\sigma_L^2)(1-\gamma)}\Big)}{\Gamma\Big(\frac{2\frac{\rho+\delta}{\gamma}+2\mu_A+(2-\gamma)\sigma_K^2-\sigma_A^2-\gamma\sigma_L^2}{(\sigma_K^2+\sigma_A^2+\sigma_L^2)(1-\gamma)}\Big)} \;<\infty\,.
\end{aligned} \tag{38}$$

For statistical and other matters, it is also reasonable to study the long-term limit of the *continuous-time-average* $\frac{1}{T}\int\limits_0^T g(k_s)ds$, where $g(\cdot)$ is some nonnegative transform-of-interest of the per capita effective capital stock. For this, we obtain the following ergodicity assertion (see Appendix A.5)

$$\begin{aligned}
&\lim_{T\to\infty}\frac{1}{T}\int\limits_0^T g(k_s)ds = \int\limits_0^\infty g(k)f_{lim}(k)\,dk \;=\; \int\limits_0^\infty g(k)f_{stea}(k)\,dk \\
&= (1-\gamma)\frac{{b_6}^{b_7}}{\Gamma(b_7)}\int\limits_0^\infty g(k)\;\exp\{-b_6k^{\gamma-1}\}\,k^{-[b_7(1-\gamma)+1]}\,dk \qquad \text{with Probability 1,}
\end{aligned} \tag{39}$$

provided that the steady-state *distribution average* (ensemble mean) $\int\limits_0^\infty g(k)\,f_{stea}(k)\,dk$ is finite. One can use (39) e.g. in connection with the verification of the consistency of some statistical estimators and tests, and also for the time-average asympotics of other, economically relevant, functionals. Of course, the most important special case is the *direct* inference upon k_t $(t \geq 0)$ which corresponds to the choice $g(x) = x$ leading to

$$\lim_{t\to\infty}\frac{1}{t}\int\limits_0^t k_s\,ds \;=\; \mu_{stea} \qquad\qquad \text{with Probability 1}$$

under the above assumption (37).

5 Simulations and the Speed of Economic Recovery

In the following, we deal with Monte Carlo simulations of our per capita effective capital stock process $(k_t)_{t\geq 0}$. To start with, we present in Table 2 an overview of concrete economy-parameter constellations recently used in some growth studies within frameworks which are similar but not entirely comparable to ours: for instance, we consider here a *continuous-time*, *fully stochastic* setup in the sense $\sigma_K \neq 0$, $\sigma_A \neq 0$, $\sigma_L \neq 0$, and we simulate $(k_t)_{t\geq 0}$ *directly* by means of the explicit *closed-form sample-path* representation formula (19) rather than by applying an appropriately adapted version of a (by its nature *indirect*) numerical Euler type method for approximating the underlying stochastic differential equation; the latter approach is typically used in contexts where no explicit sample-path formula is achieved. Working with (19), we avoid some potential pitfalls of the Euler type methods (e.g. non-convergence problems in case of wrong stepsize choices) which can appear even in the case of SDEs with very smooth drift and diffusion term, like for the prominent Cox-Ingersoll-Ross interest rate model (cf. Higham and Mao (2005)).

reference	ρ	δ	α	γ	σ_K	σ_A	μ_A	σ_L	μ_L
Acemoglu and Guerrieri (2008)	0.02	0.05	0.1–0.77	4.0					0.018
Amilon and Bermin (2003)	0.06		0.8	0.5				0.2	
Barro et al. (1995)		0.05	0.3	2.0					0.01
Bernanke and Gürkaynak (2001)	0.07–0.12		0.36	0.08–0.75					
Braun and Nakajimay (2009)	0.08		0.3	2.0					
Bucci et al. (2008)	0.04	0.05	0.33			0.0148	0.02		
Cagetti et al. (2002)	0.04	0.07	0.35	2.0		0.0192			
Cagetti and Nardi (2009)		0.06	0.33	1.5					
Caselli and Coleman (2006)			0.33						
Conesa et al. (2009)		0.0833	0.36	4.0					
Gollin et al. (2003)	0.05	0.065	0.5	1.0					
Hall (2001)		0.1							
Jensen and Richter (2007)		0.05, 0.08	0.2–0.6		0.03			0.01	0.01, 0.02
King and Rebelo (1993)		0.10	0.33	1.0–10.0					0.014
Nishide and Ohyama (2007)	0.02				0.08				
Palacios (2010)	0.001	0.05	0.36	3.5	0.12		0.017		
Posch (2009b)	0.03	0.05	0.5	1		0.02	0.01		
Pyyhtiä (2007)			0.33						
Schmitt-Grohe (2000)	0.04	0.1					0.016		
Williams (2004)	0.015	0.0517	0.35	4		0.0492	0.0176		

Table 2: Some concrete parameters in literature; "blank" spaces mean no-occurrence or lacking of comparability.

For our direct Monte Carlo simulations, we employ parameters in the range of their counterparts in Table 2. However, for the abovementioned non-comparability reasons, we take some freedom in our choices; this is also consistent with the particular goal to demonstrate the flexibility of our context for ending up with realistic economy simulations under a reasonably sized spectrum of parameter constellations (e.g. prevailing across different countries respectively different macroeconomic phases of one and the same country).

Imagine the following fictitious situation, encountered (say) at the present time $t = 0$: assume that in the near past our economy – represented by the per capita effective capital stock process $(k_t)_{t<0}$ with "old" parameters par^{old} (i.e. σ_K^{old}, σ_A^{old}, etc.) – was (approximately) in a steady-state equilibrium with old mean μ_{stea}^{old}. For transparency, let us suppose that *just before* $t = 0$ – say at time $t = 0_-$ – the economy arrived at the values K_{0_-}, A_{0_-}, L_{0_-} such that $k_{0_-} = \frac{K_{0_-}}{A_{0_-}\, L_{0_-}} = \mu_{stea}^{old}$. Furthermore, assume that at time $t = 0_-$ there was a sudden

crash in the economy environment leading to

- a destruction of the equilibrium status, here for simplicity in the sense that the random variable k_0 has – instead of the pre-crash awaited old equilibrium distribution P_{stea}^{old} – a single-valued distribution with mass concentrated at one point which (with a slight abuse of notation) is also denoted by k_0,
- a change from the old parameters par^{old} into new ones par^{new} ("parameter regime switch"), where we also allow for the special case $par^{new} = par^{old}$.

In such a context it is reasonable to ask the following question concerning the *speed of economic recovery*: how long does it take that the economy reaches (respectively arrives close to) a new equilibrium distribution P_{stea}^{new} ?

The corresponding mean of P_{stea}^{new} will be denoted by μ_{stea}^{new}. For simplicity, we confine ourselves here to the realistic special case that $\mu_{stea}^{new} = \mu_{stea}^{old}$ [11] and that $\frac{k_0 - k_{0_}}{k_{0_}} = \frac{k_0 - \mu_{stea}^{old}}{\mu_{stea}^{old}} \approx -6\%$ which means that we assume a crash of around 6% and search for the speed of recovery to return "back to the old level".

Of course, in an analogous way, one can investigate the more general question: how long does it take that an "economy of current non-equilibrium status" transits from its current value k_0 to an equilibrium distribution P_{stea}? For instance, this also applies to boom type phases in emerging economies (e.g. BRIC countries) and other, newly industrializing, countries.

Concerning this question, we carried out Monte Carlo simulations "directly" by means of our explicit closed-form sample path representation formula (19). The corresponding results will be described in the following.

Figure 1 shows the average of 200 simulated sample paths of the per capita effective capital stock process $(k_t)_{t\geq 0}$ with initial value $k_0 = 78.08$, as well as volatilities $\sigma_K^{new} = 0.08$, $\sigma_A^{new} = 0.03$, $\sigma_L^{new} = 0.02$, average growth rates $\mu_A^{new} = 0.02$, $\mu_L^{new} = 0.01$, capital share $\gamma^{new} = 0.6$, time-preference rate $\rho^{new} = 0.04$, and depreciation rate $\delta^{new} = 0.05$; according to (35), the new steady-state mean is $\mu_{stea}^{new} = 82.45$ (which is supposed to be equal to μ_{stea}^{old}) and thus we face a crash of $\frac{78.08-82.45}{82.45} \approx -5.3\%$. Notice that we have normalized the vertical axis in Figure 1 by division through $k_{0_} = 82.45$. In contrast, Figure 2 shows the analogue for the considerably lower volatilities $\sigma_K^{new} = 0.05$, $\sigma_A^{new} = 0.02$, $\sigma_L^{new} = 0.01$, lower capital share $\gamma^{new} = \frac{1}{3}$, as well as the same $\mu_A^{new} = 0.02$, $\mu_L^{new} = 0.01$, $\rho^{new} = 0.04$, $\delta^{new} = 0.05$ as above. This leads to the new steady-state mean $\mu_{stea}^{new} = 6.38$ (which is supposed to be equal to μ_{stea}^{old}); the assumption of a 6% crash amounts to the initial value $k_0 = 6.00$. Notice that we have normalized the vertical axis in Figure 2 by division through $k_{0_} = 6.38$.

[11] despite of the possibly new parameters

These two constellations provide the following Monte Carlo simulation based answers to our posed question: in Figure 1, the speed of economic recovery is about 6 years, whereas in Figure 2 the recovery speed is about 9 years.

Recall that we work here within a continuous-time *Ramsey-type* setup which is "*fully (three-dimensional) stochastic*" in the sense $\sigma_A \neq 0$, $\sigma_L \neq 0$, $\sigma_K \neq 0$, and that we have simulated the sample paths of the per capita effective capital stock process $(k_t)_{t\geq 0}$ *directly* by means of an explicit closed-form non-equilibrium sample path representation (namely, (19)). In contrast, Jensen and Richter (2007) perform similar simulations in a continuous-time lower-dimensional stochastic *Solow-Swan-type* framework, by means of an *indirect* Euler-type method. They typically take a *much larger* difference between starting value and steady-state mean than we do, leading to a – expressed in our terminology – much slower speed of recovery.

From our explicit closed-form representation formula (22), one can also simulate directly the sample paths of the capital stock process $(K_t)_{t\geq 0}$. This was carried out for yet another parameter constellation: $\sigma_K^{new} = 0.06$, $\sigma_A^{new} = 0.025$, $\sigma_L^{new} = 0.015$, $\mu_A^{new} = 0.01$, $\mu_L^{new} = 0.002$, $\gamma^{new} = 0.50$, $\rho^{new} = 0.05$, $\delta^{new} = 0.07$, which results in $\mu_{stea}^{new} = 15.92$. Furthermore, we have assumed the following crash scenario: $K_0 = 0.94 \cdot K_{0_} = 0.94 \cdot 15.92 = 14.96$ (i.e. 6% reduction of capital stock). Moreover, let us suppose that $A_0 = A_{0_} = 1$ (i.e. no change of labor-effectiveness level, which by appropriate choice of units is set to 1) as well as $L_0 = 0.99 \cdot L_{0_} = 0.99$ (i.e. 1% reduction of labor force, measured in appropriate units). This leads to $k_0 = 0.95 \cdot k_{0_} = 0.95 \cdot 15.92 = 15.12$. The corresponding average of 200 simulated sample paths of both $(K_t)_{t\geq 0}$ and $(k_t)_{t\geq 0}$ are shown in Figure 3, where for the sake of a better comparability we have normalized both processes by their pre-crash values $K_{0_}$ respectively $k_{0_}$. For the per capita effective capital stock process $(k_t)_{t\geq 0}$, the plotted simulation-runs-average (black line) indicates a recovery speed of 8 years to the steady-state equilibrium, whereas the simulation-runs-average (green/grey line) of the capital stock process $(K_t)_{t\geq 0}$ reaches his pre-crash level after about 3.8 years but (due to its nature) does not end up in a steady-state distribution and continues to grow with an average annual rate of 1.52%.

The above examples indicate in a certain sense some realistic flexibility respectively robustness of our model, which can also be seen from further Monte Carlo simulations with various different other parameter constellations; for the sake of brevity these are omitted here.

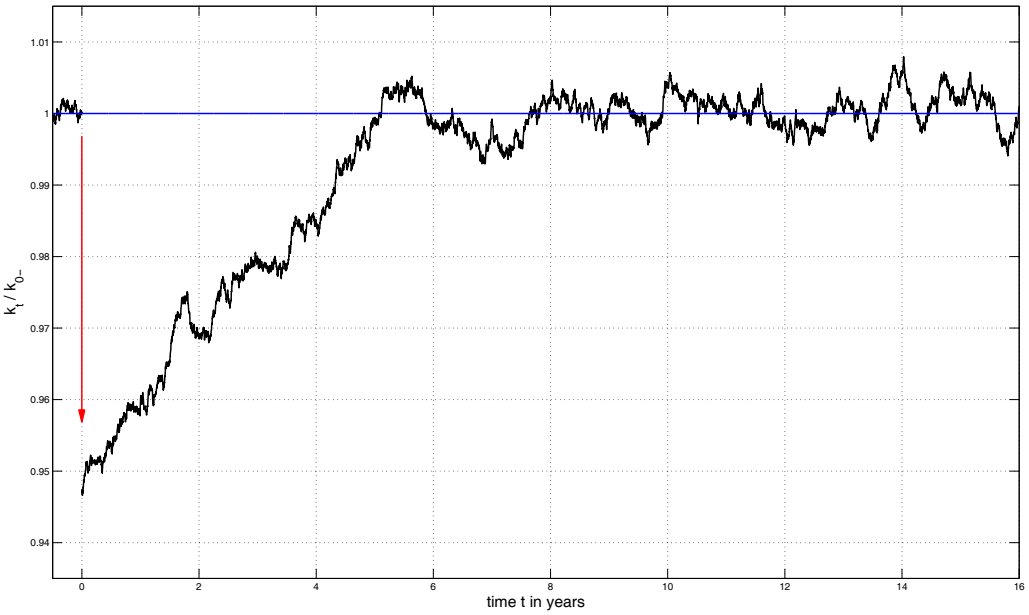

Figure 1: Plot of the average of 200 simulated sample paths of the per capita effective capital stock process $(k_t)_{t\geq 0}$ with initial value $k_0 = 78.08$, as well as steady-state mean $\mu_{stea}^{new} = 82.45$ arriving from the parameter constellation $\sigma_K^{new} = 0.08$, $\sigma_A^{new} = 0.03$, $\sigma_L^{new} = 0.02$, $\mu_A^{new} = 0.02$, $\mu_L^{new} = 0.01$, $\gamma^{new} = 0.6$, $\rho^{new} = 0.04$, $\delta^{new} = 0.05$. This means a 5.3% crash and search for the speed of a recovery to return "back to the old level".

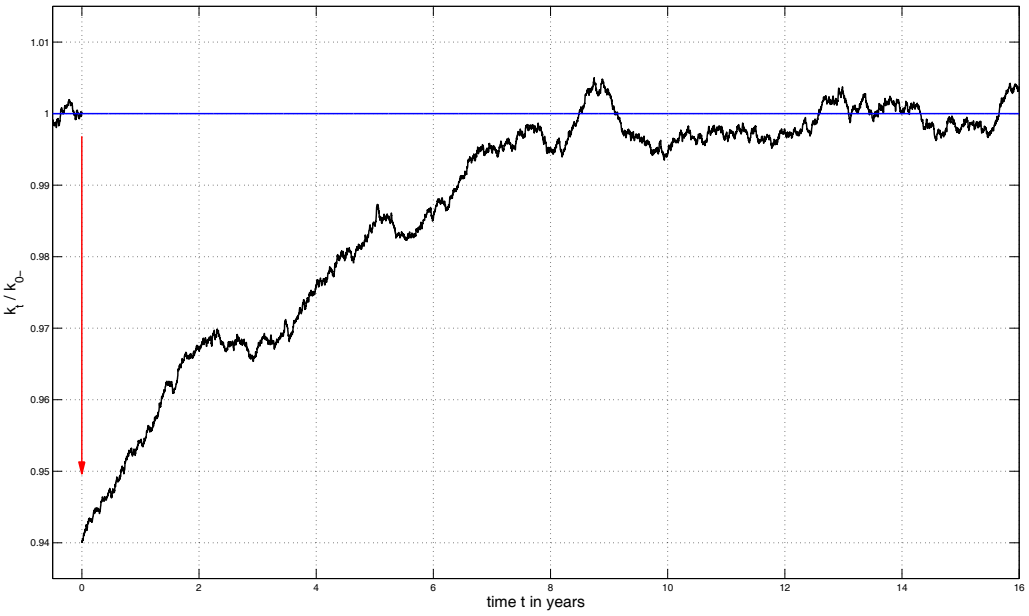

Figure 2: Plot of the average of 200 simulated sample paths of the per capita effective capital stock process $(k_t)_{t\geq 0}$ with initial value $k_0 = 6.00$, as well as steady-state mean $\mu_{stea}^{new} = 6.38$ arriving from the parameter constellation $\sigma_K^{new} = 0.05$, $\sigma_A^{new} = 0.02$, $\sigma_L^{new} = 0.01$, $\mu_A^{new} = 0.02$, $\mu_L^{new} = 0.01$, $\gamma^{new} = 0.33$, $\rho^{new} = 0.04$, $\delta^{new} = 0.05$. This means a 6.0% crash and search for the speed of a recovery to return "back to the old level".

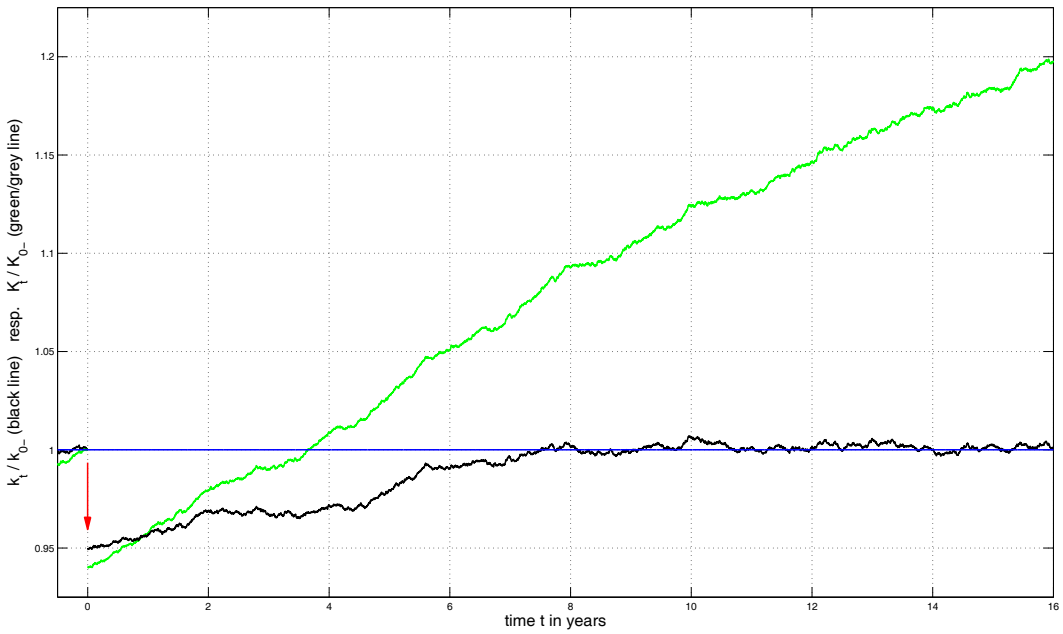

Figure 3: Plot of the average of 200 simulated sample paths of the capital stock process $(K_t)_{t\geq 0}$ (green/grey line) and of the per capita effective capital stock process $(k_t)_{t\geq 0}$ (black line), with the parameter constellation $\sigma_K^{new} = 0.06$, $\sigma_A^{new} = 0.025$, $\sigma_L^{new} = 0.015$, $\mu_A^{new} = 0.01$, $\mu_L^{new} = 0.002$, $\gamma^{new} = 0.50$, $\rho^{new} = 0.05$, $\delta^{new} = 0.07$.

Let us finally mention that for the economy-value processes one can also obtain some statistical decision sensitivity results along the lines of Stummer and Vajda (2007). For the sake of brevity, this will appear in a forthcoming paper.

A Appendices

A.1 Derivation of the optimal consumption strategy

As usual, we first derive formally the Hamilton-Jacobi-Bellman (HJB) equation which corresponds to the Optimization Problem 2.1 for the special case $s = 0$ (cf. (12)). Let us first observe that for any (admissible) consumption strategy $c_t = c(X_t)$ the three-dimensional economy-value process $(X_t)_{t\geq 0}$ defined by $X_t = (K_t, A_t, L_t)$ is per assumption a unique strong solution of the system of the three SDEs (10), (2), (3) where the latter two are uncoupled linear (and thus straightforwardly treatable) SDEs. By employing the standard stochastic-control-theoretic techniques (see e.g. Øksendal (2007, Chapter 11), Øksendal and Sulem (2007, Chapter 3)), the corresponding HJB-equation has the form – in terms of $X = (K, A, L) > 0$ –

$$\sup_{v\in[0,\infty)} \Big\{ \frac{v^{1-\gamma}-1}{1-\gamma} L - \rho W(X) + [K^\gamma (AL)^{1-\gamma} - \delta K - Lv] \frac{\partial W(X)}{\partial K} + \mu_A A \frac{\partial W(X)}{\partial A}$$
$$+ \mu_L L \frac{\partial W(X)}{\partial L} + \tfrac{1}{2}\sigma_K^2 K^2 \frac{\partial^2 W(X)}{\partial K^2} + \tfrac{1}{2}\sigma_A^2 A^2 \frac{\partial^2 W(X)}{\partial A^2} + \tfrac{1}{2}\sigma_L^2 L^2 \frac{\partial^2 W(X)}{\partial L^2} \Big\} = 0 \ . \quad (40)$$

By checking the first-order and second-order conditions one can immediately see that the maximization (40) is uniquely solved by

$$v \;=\; \left(\frac{\partial W(X)}{\partial K}\right)^{-\frac{1}{\gamma}} \;=:\; v^{max}(X)\;. \tag{41}$$

By plugging this into (40) we obtain

$$\begin{aligned}&\frac{\left(\frac{\partial W(X)}{\partial K}\right)^{-\frac{1-\gamma}{\gamma}}-1}{1-\gamma}\,L-\rho W(X)+\left[K^{\gamma}(AL)^{1-\gamma}-\delta K-L\cdot\left(\frac{\partial W(X)}{\partial K}\right)^{-\frac{1}{\gamma}}\right]\frac{\partial W(X)}{\partial K}\\&+\mu_A A\frac{\partial W(X)}{\partial A}+\mu_L L\frac{\partial W(X)}{\partial L}+\tfrac{1}{2}\sigma_K^2K^2\frac{\partial^2 W(X)}{\partial K^2}+\tfrac{1}{2}\sigma_A^2A^2\frac{\partial^2 W(X)}{\partial A^2}+\tfrac{1}{2}\sigma_L^2L^2\frac{\partial^2 W(X)}{\partial L^2}=0\;.\end{aligned} \tag{42}$$

As a potential solution candidate, let us try the the educated guess

$$W(X):=b_8K^{1-\gamma}L^{\gamma}+b_9A^{1-\gamma}L+b_{10}L \tag{43}$$

for some constants $b_8, b_9, b_{10}\in\mathbb{R}$, which transforms (42) into

$$\begin{aligned}&\tfrac{[(1-\gamma)b_8]^{-\frac{1-\gamma}{\gamma}}(\frac{K}{L})^{1-\gamma}-1}{1-\gamma}\,L-\rho[b_8K^{1-\gamma}L^{\gamma}+b_9A^{1-\gamma}L+b_{10}L]\\&+\big[K^{\gamma}(AL)^{1-\gamma}-\delta K-L[(1-\gamma)b_8]^{-\frac{1}{\gamma}}\tfrac{K}{L}\big]b_8(1-\gamma)K^{-\gamma}L^{\gamma}+\mu_AAb_9(1-\gamma)A^{-\gamma}L\\&+\mu_LL[b_8\gamma K^{1-\gamma}L^{-(1-\gamma)}+b_9A^{1-\gamma}+b_{10}]+\tfrac{1}{2}\sigma_K^2K^2[-b_8(1-\gamma)\gamma K^{-(1+\gamma)}L^{\gamma}]\\&+\tfrac{1}{2}\sigma_A^2A^2[-b_9(1-\gamma)\gamma A^{-(1+\gamma)}L]+\tfrac{1}{2}\sigma_L^2L^2[-b_8(1-\gamma)\gamma K^{1-\gamma}L^{-(2-\gamma)}]=0\quad\text{for all } K,A,L>0\,,\end{aligned}$$

which is equivalent to

$$\begin{aligned}&K^{1-\gamma}L^{\gamma}b_8\Big\{\gamma[(1-\gamma)b_8]^{-\frac{1}{\gamma}}-\rho-\delta(1-\gamma)+\mu_L\gamma-\tfrac{1}{2}\sigma_K^2(1-\gamma)\gamma-\tfrac{1}{2}\sigma_L^2(1-\gamma)\gamma\Big\}\\&+A^{1-\gamma}L\Big\{-\rho b_9+(1-\gamma)b_8+\mu_A(1-\gamma)b_9+\mu_Lb_9-\tfrac{1}{2}\sigma_A^2(1-\gamma)\gamma b_9\Big\}\\&+L\Big\{-\tfrac{1}{(1-\gamma)}-(\rho-\mu_L)b_{10}\Big\}=0\qquad\text{for all } K,A,L>0\,.\end{aligned} \tag{44}$$

But (44) holds if and only if the following three equations for (b_8, b_9, b_{10}) hold:

$$b_1 \;:=\; [(1-\gamma)b_8]^{-\frac{1}{\gamma}}=\tfrac{1}{\gamma}\big[\rho+\delta(1-\gamma)-\mu_L\gamma+\tfrac{1}{2}(1-\gamma)\gamma(\sigma_K^2+\sigma_L^2)\big]\,, \tag{45}$$

$$b_9=\frac{b_8(1-\gamma)}{\rho-\mu_A(1-\gamma)-\mu_L+\frac{1}{2}\sigma_A^2(1-\gamma)\gamma}\,, \tag{46}$$

$$b_{10}=-\frac{1}{(1-\gamma)(\rho-\mu_L)}\,. \tag{47}$$

Notice that by assumption (7) the right-hand side of (45) as well as the denominators in (46), (47) are all strictly positive. Consequently, there holds $b_8, b_9, b_{10} > 0$ which implies $W(X) > 0$. Furthermore, the function $W(\cdot)$ is twice continuously differentiable on $(0,\infty)\times(0,\infty)\times(0,\infty)$, continuous on $[0,\infty)\times[0,\infty)\times[0,\infty)$ as well as linearly bounded; with respect to the component K, the function $W(\cdot)$ is strictly increasing and strictly concave. Altogether,

one gets $W(X)$ explicitly from (43) together with the unique solution (b_8, b_9, b_{10}) of (45), (46), (47). From (41), (43) and (45) we obtain

$$v^{max}(X) = v^{max}(K, A, L) = b_1 \frac{K}{L} \ > 0 \tag{48}$$

which coincides with (13).

As usual, the classical solution $W(\cdot)$ of the HJB (40) is a canditate for the desired optimal expected lifetime utility of consumption $V(0, \cdot)$ given in (11), and $v^{max}(\cdot)$ a canditate for the corresponding optimal consumption strategy $c^{opt}(\cdot)$. However, in order to prove that this is indeed the solution of the Optimization Problem 2.1 with $s = 0$ (cf. (12)) one has to run through a standard verification procedure. This involves in principle (properties of) the corresponding optimal economy-value process $(X_t)_{t\geq 0}$ for which we shall derive the explict closed form in the following two Appendices A.2 and A.3. Thereafter, in Appendix A.4, we shall deal with the abovementioned verification procedure including in particular the transversality condition.

A.2 Solution of the SDE (17)

In order to obatin a strong solution of the SDE (17), we shall use the following procedure which for potential future research we develop in general terms and which is an extension of a method e.g. presented in Øksendal (2007, Chapter 5). Consider a one-dimensional SDE of the form

$$\begin{aligned} dX_t &= f(t, X_t)dt + X_t \left[\sigma_1(t)dV_t^{(1)} + \sigma_2(t)dV_t^{(2)} + \sigma_3(t)dV_t^{(3)}\right] \\ &= f(t, X_t)dt + X_t\, \sigma(t)^{tr} \cdot dV_t\,, \qquad\qquad X_0 = x\,,\ t \geq 0\,, \end{aligned} \tag{49}$$

where $f(\cdot,\cdot)$, $\sigma_1(\cdot), \sigma_2(\cdot), \sigma_3(\cdot)$ are deterministic continuous real-valued functions and $V_t = (V_t^{(1)}, V_t^{(2)}, V_t^{(3)})^{tr}$ is a three-dimensional Gauss process[12] with continuous[13] sample paths, independent increments, $V_0 = 0$, $\mathrm{E}[V_t] = 0$, and absolutely continuous variance function $\left\{\mathrm{E}[V_t\, V_t^{tr}]\right\}_{t\geq 0}$ (see e.g. Arnold (1974)). Thus, one can represent V_t as a stochastic integral[14]

$$V_t = \int_0^t G(s)\, dW_s\,, \tag{50}$$

where $\{W_t\}_{t\geq 0}$ a three-dimensional standard Brownian motion, and $G(\cdot)$ is a deterministic function taking values in the space of all (3×3)-matrices such that $\int_0^t |G(s)|^2 ds < \infty$ for all $t \geq 0$ (cf. Arnold (1974)). Hence, $\{V_t\}_{t\geq 0}$ can be interpreted as a "straightforward" time-

[12] here, V_t (and thus X_t) is interpreted as a column vector, and tr indicates henceforth the transpose of a vector respectively a matrix

[13] (with probability 1)

[14] of Ito type

inhomogeneous generalization of a Brownian motion. The following assertion describes how to construct a SDE solution from a (pathwise) solution of a – generally easier to handle – *ordinary* differential equation (ODE)[15]:

Lemma A.1. *Suppose that $G(\cdot)$ is continuous and define the auxiliary stochastic process $(F_t(\omega))_{t\geq 0}$ by*

$$F_t(\omega) \;:=\; \exp\Big\{ -\int_0^t \sigma_1(s)dV_s^1(\omega) - \int_0^t \sigma_2(s)dV_s^2(\omega) - \int_0^t \sigma_3(s)dV_s^3(\omega)$$
$$+\tfrac{1}{2}\int_0^t \Big[\big(\sigma_1(s),\sigma_2(s),\sigma_3(s)\big)\cdot G(s)\,G(s)^{tr}\cdot \begin{pmatrix}\sigma_1(s)\\ \sigma_2(s)\\ \sigma_3(s)\end{pmatrix}\Big]\,ds\Big\} > 0 \quad \textit{with probability 1.}$$

Then the following holds: if there exists a stochastic process $(Y_t(\omega))_{t\geq 0}$ such that for each fixed $\omega\in\Omega$ the corresponding (deterministic-function type) sample path $t\mapsto Y_t(\omega)$ is a solution of the ODE

$$\frac{dY_t(\omega)}{dt} = F_t(\omega)\, f\big(t, F_t^{-1}(\omega)Y_t(\omega)\big)\,, \quad Y_0(\omega)=x\,, \tag{51}$$

then the stochastic process $(X_t(\omega))_{t\geq 0}$ defined by $X_t(\omega):=F_t^{-1}(\omega)Y_t(\omega)$ is a strong solution of the SDE (49).

Proof. One can represent $F_t = g(t,Z_t)$, where

$$g(t,z):=\exp\big\{-z+\tfrac{1}{2}\int_0^t \widetilde{\sigma}(s)ds\big\} \quad \text{with}$$
$$\widetilde{\sigma}(s) = \big(\sigma_1(s),\sigma_2(s),\sigma_3(s)\big)\cdot G(s)\,G(s)^{tr}\cdot \begin{pmatrix}\sigma_1(s)\\ \sigma_2(s)\\ \sigma_3(s)\end{pmatrix}\;,$$

and $Z_t := \int_0^t \sigma(s)^{tr}dV_s = \int_0^t \sigma(s)^{tr}G_s\,dW_s$. Thus, by Ito's formula we obtain

$$\begin{aligned} dF_t &= \frac{\partial g}{\partial t}(t,Z_t)dt + \frac{\partial g}{\partial z}(t,Z_t)dZ_t + \tfrac{1}{2}\frac{\partial^2 g}{\partial z^2}(t,Z_t)(dZ_t)^2 \\ &= \tfrac{1}{2}F_t\widetilde{\sigma}(t)dt - F_t\sigma(t)^{tr}G(t)\,dW_t + \tfrac{1}{2}F_t\left(\sigma(t)^{tr}G(t)\,dW_t\right)^2 \\ &= F_t\widetilde{\sigma}(t)dt - F_t\sigma(t)^{tr}G(t)\,dW_t \end{aligned} \tag{52}$$

which by another application of Ito's formula leads to

$$\begin{aligned} d\left(F_t^{-1}\right) &= -F_t^{-2}\,dF_t + F_t^{-3}\,(dF_t)^2 \;=\; -F_t^{-1}\,\widetilde{\sigma}(t)\,dt \;+\; F_t^{-1}\,\sigma(t)^{tr}\,G(t)\,dW_t \;+\; F_t^{-1}\,\widetilde{\sigma}(t)\,dt \\ &= F_t^{-1}\,\sigma(t)^{tr}\,G(t)\,dW_t\;. \end{aligned}$$

From the appropriate two-dimensional version of Ito's formula and (51) we get

[15] for the sake of transparancy, we exceptionally quote the involved sample point ω in the formulation (but not the proof) of Lemma A.1, and in its concrete application below

$$
\begin{aligned}
dX_t &= d\left(F_t^{-1}\,Y_t\right) = F_t^{-1}\,dY_t + Y_t\,d\left(F_t^{-1}\right) + d\left(F_t^{-1}\right)\,dY_t \\
&= F_t^{-1}\,F_t\;f\left(t, F_t^{-1}\,Y_t\right)\,dt \;+\; Y_t\;F_t^{-1}\,\sigma(t)^{tr}\,G(t)\,dW_t \;+\; 0
\end{aligned}
$$

which leads immediately to the desired SDE (49) with initial condition $X_0 = Y_0 = x$. □

Let us now apply the general Lemma A.1 in order to derive a solution of the special case

$$
dk_t = \left[{k_t}^{\gamma} - b_2 k_t\right]dt + k_t\left[\sigma_K dB_t^K - \sigma_A dB_t^A - \sigma_L dB_t^L\right], \qquad k_0 > 0, \;\; t \geq 0. \qquad \text{(cf. (17))}
$$

Accordingly, we set $f(t,k_t) := {k_t}^{\gamma} - b_2 k_t$, $\sigma_1(t) := \sigma_K$, $\sigma_2(t) := -\sigma_A$, $\sigma_3(t) := -\sigma_L$, $W_t := \left(B_t^K, B_t^A, B_t^L\right)^{tr}$ and $G(t) := I_3$ with three-dimensional unit matrix I_3. Hence, there holds $\widetilde{\sigma}(s) \equiv \sigma_K^2 + \sigma_A^2 + \sigma_L^2$ as well as $F_t(\omega) = \exp\left\{-\sigma_K B_t^K(\omega) + \sigma_A B_t^A(\omega) + \sigma_L B_t^L(\omega) + \frac{1}{2}(\sigma_K^2 + \sigma_A^2 + \sigma_L^2)t\right\}$. For fixed $\omega \in \Omega$ we search for the function $t \mapsto Y_t(\omega)$ which solves uniquely the ODE

$$
\frac{dY_t(\omega)}{dt} = F_t(\omega)\left[\left(\frac{Y_t(\omega)}{F_t(\omega)}\right)^{\gamma} - b_2\,\frac{Y_t(\omega)}{F_t(\omega)}\right] = F_t(\omega)^{1-\gamma}\,Y_t(\omega)^{\gamma} - b_2 Y_t(\omega), \quad Y_0 = k_0, \qquad (53)
$$

which can be found by straightforward ODE-techniques as

$$
Y_t(\omega) = \mathrm{e}^{-b_2 t}\left[\int_0^t (1-\gamma)F_s(\omega)^{1-\gamma}\,\mathrm{e}^{b_2(1-\gamma)s}\,ds + {k_0}^{1-\gamma}\right]^{\frac{1}{1-\gamma}}.
$$

Thus, by Lemma A.1, the process $(k_t)_{t\geq 0}$ defined by

$$
\begin{aligned}
k_t(\omega) = F_t^{-1}(\omega)Y_t(\omega) &= \mathrm{e}^{\sigma_K B_t^K(\omega) - \sigma_A B_t^A(\omega) - \sigma_L B_t^L(\omega) - \frac{1}{2}b_3 t}\;. \\
&\cdot \left[(1-\gamma)\int_0^t \mathrm{e}^{(1-\gamma)\left[-\sigma_K B_s^K(\omega) + \sigma_A B_s^A(\omega) + \sigma_L B_s^L(\omega) + \frac{1}{2}b_3 s\right]}\,ds + {k_0}^{1-\gamma}\right]^{\frac{1}{1-\gamma}}, \qquad \text{(cf. (19))}
\end{aligned}
$$

$$
\text{with} \;\; b_3 := \sigma_K^2 + \sigma_A^2 + \sigma_L^2 + 2b_2 \underset{(18)}{=} 2\frac{\rho+\delta}{\gamma} + 2\mu_A + (2-\gamma)\sigma_K^2 - \sigma_A^2 - \gamma\sigma_L^2\;, \qquad \text{(cf. (20))}
$$

is a strong solution of the SDE (17). It is straightforward to see – directly respectively from the belowmentioned formulae (61), (62) below together with Prop. 5.22 of Karatzas and Shreve (1991) – that the solution $(k_t)_{t\geq 0}$ does not explode in finite time with probability 1. In order to prove the strong uniqueness of solutions of (17), one can apply e.g. the corresponding general results of Engelbert and Schmidt (1991, Section 4.5).

A.3 Transformation of (17) into (14)

By using the appropriate three-dimensional version of Ito's formula together with the SDEs (2), (3), (17) as well as the parameter definition (18), one can calculate

$$
\begin{aligned}
dK_t &= d(k_t A_t L_t) \qquad (54)\\
&= A_t L_t dk_t + A_t k_t dL_t + L_t k_t dA_t + k_t dA_t dL_t + A_t dk_t dL_t + L_t dk_t dA_t + dk_t dA_t dL_t\\
&= A_t L_t \left\{ \left[{k_t}^{\gamma} - b_2 k_t \right] dt + k_t \left[\sigma_K dB_t^K - \sigma_A dB_t^A - \sigma_L dB_t^L \right] \right\}\\
&\quad + A_t k_t \left\{ \mu_L L_t dt + \sigma_L L_t dB_t^L \right\} + L_t k_t \left\{ \mu_A A_t dt + \sigma_A A_t dB_t^A \right\} + k_t \cdot 0\\
&\quad + A_t \cdot \left\{ -\sigma_L^2 L_t k_t dt \right\} + L_t \cdot \left\{ -\sigma_A^2 A_t k_t dt \right\} + 0\\
&= \left[K_t^{\gamma} (A_t L_t)^{1-\gamma} - (\delta + b_1) K_t \right] dt + \sigma_K K_t dB_t^K
\end{aligned}
$$

which means that $(K_t)_{t\geq 0}$ defined by (54) (with arbitrary initial value $K_0 > 0$) is a strong solution of the SDE (14). The corresponding strong uniqueness can be shown as follows: let $(\widetilde{K}_t)_{t\geq 0}$ be another strong solution of (14) with initial value $\widetilde{K}_0 = K_0$. Then by Ito's formula

$$
\begin{aligned}
d\widetilde{k}_t &:= d\frac{\widetilde{K}_t}{A_t L_t} \qquad (55)\\
&= \frac{1}{A_t L_t} \left\{ \left[{\widetilde{K}_t}^{\gamma} (A_t L_t)^{1-\gamma} - (\delta + b_1) \widetilde{K}_t \right] dt + \sigma_K \widetilde{K}_t dB_t^K \right\} - \frac{\widetilde{K}_t}{A_t^2 L_t} \left[\mu_A A_t dt + \sigma_A A_t dB_t^A \right]\\
&\quad - \frac{\widetilde{K}_t}{A_t L_t^2} \left[\mu_L L_t dt + \sigma_L L_t dB_t^L \right] + \frac{\widetilde{K}_t}{A_t^3 L_t} \sigma_A^2 A_t^2 dt + \frac{\widetilde{K}_t}{A_t L_t^3} \sigma_L^2 L_t^2 dt\\
&= \left[{\widetilde{k}_t}^{\gamma} - b_2 \widetilde{k}_t \right] dt + \widetilde{k}_t \left[\sigma_K dB_t^K - \sigma_A dB_t^A - \sigma_L dB_t^L \right] \ ,
\end{aligned}
$$

which shows that $(\widetilde{k}_t)_{t\geq 0}$ defined by (55) is a strong solution of the SDE (17) with initial value $\widetilde{k}_0 = k_0$. Hence, by the strong uniqueness of (17) established in Appendix A.2 above, we have with probability 1 that $\widetilde{k}_t = k_t$ for all $t \geq 0$. Thus, with probability 1 there holds $\widetilde{K}_t = K_t$ for all $t \geq 0$, which is just the desired strong uniqueness for (14).

A.4 Verification

Recall from Appendix A.1 that the function $W(\cdot)$ defined by (43) is a canditate for the desired optimal expected lifetime utility of consumption $V(0,\cdot)$ given in (11), and $v^{max}(\cdot)$ (cf.(48)) a candidate for the corresponding optimal consumption strategy $c^{opt}(\cdot)$. To show that this is indeed the desired solution of Optimization Problem 2.1 with $s = 0$ (cf. (12)), one can employ the usual (adequately adapted) standard verification procedure, see e.g. Øksendal (2007, Chapter 11), Øksendal and Sulem (2007, Chapter 3). As a part of this, one can conclude from the above considerations that $v^{max}(\cdot)$ is admissible, i.e. $(v^{max}(X_t))_{t\geq 0} \in \mathcal{C}(0,x)$. Furthermore, the transversality condition

$$
\lim_{t\to\infty} \mathrm{e}^{-\rho t} \, \mathrm{E}\big[W(X_t)\big] = 0 \qquad (56)
$$

holds, which can be seen as follows:

$$
\begin{aligned}
&\mathrm{e}^{-\rho t}\,\mathrm{E}\big[W(X_t)\big] = \mathrm{e}^{-\rho t}\,\mathrm{E}\big[b_8 K_t^{1-\gamma} L_t^{\gamma} + b_9 A_t^{1-\gamma} L_t + b_{10} L_t\big] \\
&= (1-\gamma) b_8 A_0^{1-\gamma} L_0 \int_0^t \mathrm{E}\big[\exp\big\{-\big(b_1 + \tfrac{1}{2}(1-\gamma)^2\sigma_K^2 + \tfrac{1}{2}\gamma^2\sigma_L^2\big)t + \tfrac{1}{2}(1-\gamma) b_3 s \\
&\quad + (1-\gamma)\sigma_K (B_t^K - B_s^K) + (1-\gamma)\sigma_A B_s^A + \gamma\sigma_L (B_t^L - B_s^L) + \sigma_L B_s^L\big\}\big]\,ds \\
&\quad + b_8 K_0^{1-\gamma} L_0^{\gamma}\,\mathrm{E}\big[\exp\big\{-\big(b_1 + \tfrac{1}{2}(1-\gamma)^2\sigma_K^2 + \tfrac{1}{2}\gamma^2\sigma_L^2\big)t + (1-\gamma)\sigma_K B_t^K + \gamma\sigma_L B_t^L\big\}\big] \\
&\quad + b_9 A_0^{1-\gamma} L_0\,\mathrm{E}\big[\exp\big\{\big((1-\gamma)\mu_A + \mu_L - \rho - \tfrac{1}{2}(1-\gamma)\sigma_A^2 - \tfrac{1}{2}\sigma_L^2\big)t + (1-\gamma)\sigma_A B_t^A + \sigma_L B_t^L\big\}\big] \\
&\quad + b_{10} L_0\,\mathrm{E}\big[\exp\big\{\big(\mu_L - \rho - \tfrac{1}{2}\sigma_L^2\big)t + \sigma_L B_t^L\big\}\big] \\
&\quad = (1-\gamma) b_8 A_0^{1-\gamma} L_0 \int_0^t \mathrm{e}^{-b_1 t}\,\mathrm{e}^{b_{11} s}\,ds + b_8 K_0^{1-\gamma} L_0^{\gamma}\,\mathrm{e}^{-b_1 t} + b_9 A_0^{1-\gamma} L_0\,\mathrm{e}^{(b_{11}-b_1)t} + b_{10} L_0\,\mathrm{e}^{(\mu_L-\rho)t}
\end{aligned}
\tag{57}
$$

with $b_{11} := (1-\gamma)\frac{\rho+\delta}{\gamma} + (1-\gamma)\mu_A + \frac{1}{2}(1-\gamma)\sigma_K^2 - \frac{1}{2}(1-\gamma)\gamma\sigma_A^2 + \frac{1}{2}(1-\gamma)\sigma_L^2$.

If $b_{11} = 0$ then $\int_0^t \mathrm{e}^{-b_1 t}\,\mathrm{e}^{b_{11} s}\,ds = t\,\mathrm{e}^{-b_1 t} \xrightarrow[t\to\infty]{} 0$ (recall that $b_1 > 0$ by (7) and (45)). If $b_{11} \neq 0$ then one gets $\int_0^t \mathrm{e}^{-b_1 t}\,\mathrm{e}^{b_{11} s}\,ds = \frac{1}{b_{11}}\big(\mathrm{e}^{(b_{11}-b_1)t} - \mathrm{e}^{-b_1 t}\big) \xrightarrow[t\to\infty]{} 0$ since $b_1 > 0$ and $b_{11} - b_1 = (1-\gamma)\mu_A + \mu_L - \rho - \frac{1}{2}(1-\gamma)\gamma\sigma_A^2 < 0$ (cf. (7)). Because of $b_1 > 0$, $b_{11} - b_1 < 0$ and $\mu_L - \rho < 0$ (cf. (7)), also the last three terms in (57) converge to 0 as t tends to ∞. Thus, we have verified (56).

A.5 Derivation of the limit distribution, steady-state distribution and ergodicity assertion (39)

In this section we deduce the long-term behaviour of the non-equilibrium per capita effective capital stock process $(k_t)_{t\geq 0}$ with dynamics

$$
dk_t = \big[k_t{}^{\gamma} - b_2 k_t\big]dt + k_t\,\big[\sigma_K dB_t^K - \sigma_A dB_t^A - \sigma_L dB_t^L\big]\ , \qquad \text{(cf. (17))}
$$

by applying (an adaption of) the corresponding general "ergodic" theory of one-dimensional stochastic differential equations, see e.g. Karatzas and Shreve (1991), Kallenberg (2002). Notice that in (17) both the drift term and the diffusion term are bounded on compact subintervals of the domain $I = (0,\infty)$. By this and the nondegeneracy $\sigma_K^2 + \sigma_A^2 + \sigma_L^2 > 0$, all the following quantities are well defined. Let us first examine the scale function

$$
\begin{aligned}
s(k) \quad &:= \quad \int_1^k \exp\Big\{-2\int_1^{\xi} \tfrac{y^{\gamma} - b_2 y}{(\sigma_K^2+\sigma_A^2+\sigma_L^2)\cdot y^2}\,dy\Big\}\,d\xi && (58)\\
&= \quad -\tfrac{1}{1-\gamma}\,\mathrm{e}^{-b_6}\int_1^{k^{\gamma-1}} \exp\big\{b_6\zeta - b_{12}\ln\zeta\big\}\,d\zeta\ , \qquad k\in(0,\infty), && (59)\\
\text{with}\quad & b_6 := \tfrac{2}{(\sigma_K^2+\sigma_A^2+\sigma_L^2)\cdot(1-\gamma)} > 0 \qquad \text{(cf. (32))} \\
\text{and}\quad & b_{12} := \tfrac{2b_2}{(\sigma_K^2+\sigma_A^2+\sigma_L^2)\cdot(1-\gamma)} + \tfrac{1}{1-\gamma} + 1 = \tfrac{b_3}{(\sigma_K^2+\sigma_A^2+\sigma_L^2)\cdot(1-\gamma)} + 1\,, && (60)
\end{aligned}
$$

where b_3 is given by (20). It is easy to see that

$$\lim_{k\to 0+} s(k) = -\infty \tag{61}$$

holds. Moreover, under the assumption $b_{12} > 1$ which is equivalent to

$$b_3 > 0\,, \qquad\qquad \text{(cf. (30))}$$

one gets

$$\lim_{k\to\infty} s(k) = +\infty\,. \tag{62}$$

Another important tool in this context is the speed measure density

$$m(k) \;:=\; \frac{2}{s'(k)\cdot(\sigma_K^2+\sigma_A^2+\sigma_L^2)\cdot k^2} \;=\; \frac{2}{\exp\Big\{b_6\,[k^{\gamma-1}+b_2(1-\gamma)\ln k-1]\Big\}\cdot(\sigma_K^2+\sigma_A^2+\sigma_L^2)\cdot k^2} \tag{63}$$

and the corresponding speed measure defined by $\widetilde{m}(A) = \int\limits_A m(k)dk$ for all Borel sets A in $(0,\infty)$. By means of the well-known gamma function $\Gamma(\cdot)$ one immediately arrives at

$$\widetilde{m}((0,\infty)) \;=\; b_6\,\mathrm{e}^{b_6}\int\limits_0^\infty \mathrm{e}^{-b_6 y}\,y^{b_7-1}dy = {b_6}^{1-b_7}\,\mathrm{e}^{b_6}\,\Gamma(b_7) < \infty\,. \tag{64}$$

Since by (23), every k_t $(t \geq 0)$ has an absolutely continuous distribution, one can apply a result of Pollak and Siegmund (1985) (see also Karatzas and Shreve, 1991, p. 352) to obtain the desired limit distribution function (31) by

$$\begin{aligned} F_{lim}(z) \;&:=\; \lim_{t\to\infty} \mathrm{P}^{0,k_0}\big[k_t \leq z\big] = \frac{\widetilde{m}\big((0,z)\big)}{\widetilde{m}((0,\infty))} \\ &=\; \frac{\int\limits_0^z \frac{2}{k^2\cdot(\sigma_K^2+\sigma_A^2+\sigma_L^2)\cdot\exp\{b_6[k^{\gamma-1}+b_2(1-\gamma)\ln k-1]\}}\,dk}{{b_6}^{1-b_7}\,\mathrm{e}^{b_6}\,\Gamma(b_7)} \\ &=\; (1-\gamma)\frac{{b_6}^{b_7}}{\Gamma(b_7)}\int\limits_0^z \exp\big\{-b_6 k^{\gamma-1}\big\}k^{-[b_7(1-\gamma)+1]}\,dk\,, \qquad z\in(0,\infty)\,, \end{aligned} \tag{65}$$

where we have used (63), (64), (32) and (33). Finally, the ergodicity assertion (39) as well as the equality between steady-state and limit distribution (i.e. $P_{stea} = P_{lim}$) follow by standard techniques, see e.g. Kallenberg (2002, Chapter 23).

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