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Abstract

There are several procedures to construct a skewed distribution. One of these procedures splits the value of a parameter of scale for the two halfs of a symmetric distribution. Fechner proposed this procedure in his famous book "Kollektivmaßlehre (1897), p. 295ff.". A similar proposal comes from Fernández et al. (1995). We consider the very general approach from Arellano-Valle et al. (2005) of splitting a scale parameter and show that this technique of generating skewed distributions incorporates a well-defined parameter of skewness. It is well-defined in the sense that the parameter of skewness is compatible with the ordering $\preceq_2$ of van Zwet (1964) which is the strongest ordering in the hierarchy of orderings discussed by Oja (1981). For this family of skewed distributions it will be shown that the measure proposed by Arnold & Groeneveld (1995) is a measure of skewness in the sense of Oja (1981). In the special case considered by Fechner (1897) this measure and the skewness parameter coincide.

Keywords: Skewness; skewness to the right; skewness ordering, measure of skewness

1 Introduction

Starting with a symmetric density f Arellano-Valle et al. (2005) introduce an asymmetric distribution

$$f(x;\gamma) = \frac{2}{a(\gamma)+b(\gamma)}\left(f\left(\frac{x}{a(\gamma)}\right)I(x<0) + f\left(\frac{x}{b(\gamma)}\right)I(x\geq 0)\right) \tag{1}$$

where $a(.)$, $b(.)$ are known positive functions with domain Γ. In the following we call $a(.)$ and $b(.)$ skewness functions. This is a general kind of generating skewness by splitting a scale parameter for the negative and positive half of a distribution that has several well-known special cases. In the following we call this family the AGQ family of skewed distributions.

Setting $a(\gamma) = \gamma$ and $b(\gamma) = 1/\gamma$ for $\Gamma = \mathbb{R}^+$ we get an asymmetric density discussed by Fernández et al. (1995) and Theodossiou (1998). Another choice is $a(\gamma) = 1+\gamma$ and $b(\gamma) = 1-\gamma$ for $\gamma \in \Gamma = (-1, 1)$. If f is the normal density we get an asymmetric density already considered by Fechner (1897), p. 295ff. in his famous book "Kollektivmaßlehre".

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Arellano-Valle et al. (2005) discuss the statistical properties of (1) for different choices of $a(.)$, $b(.)$ and $f(.)$. Furthermore, they discuss inferential aspects like parameter estimation by the method of moments and by maximum likelihood. For (1) Cassart et al. (2008) construct optimal tests on skewness.

Arellano-Valle et al. (2005, p. 429) or Cassart et al. (2008, p. 2500) call γ an "asymmetry" or a "skewness" parameter. But, due to the work of van Zwet (1964), Oja (1981), Balanda & McGillivray (1990) and others there exist conditions a parameter has to satisfy to be a skewness parameter. Especially, a skewness parameter must hold a specified ordering of skewness. This still has to be proven for the parameter γ in (1).

Klein & Fischer (2006) show for the special choice $a(\gamma) = \gamma$ and $b(\gamma) = 1/\gamma$ (see Fernández et al. (1995)) that γ is a skewness parameter. In detail, they show that $f(x;\gamma)$ is skewed to the right for $\gamma < 1$ and that under some conditions of regularity concerning the symmetric density f γ holds the strong ordering of skewness of van Zwet (1964). This ordering concerns the convexity of

$$\Lambda(x;\gamma_1,\gamma_2)) = F^{-1}(F(.;\gamma_1);\gamma_2) \;\; x \in supp(F(.;\gamma_1)) \tag{2}$$

with the cumulative distribution function $F(.;\gamma)$ and the quantile function $F^{-1}(.;\gamma)$ corresponding to (1).

In this paper we will generalize the results of Klein & Fischer (2006). We show that $\Lambda(.;\gamma_1,\gamma_2)$ is convex or concave for positive, monotone increasing functions $a(.)$ and positive, monotone decreasing functions $b(.)$ with existing derivative $a'(.)$ and $b'(.)$. The conditions of regularity on f are the same as in Klein & Fischer (2006).

The paper is organized in the following way. In section 2 we derive the distribution function, the quantile function, the score function and a function ϕ that is important to show the convexity of (2). Section 3 contains the central result that the AGQ family is a family of skewed distributions in the sense of van Zwet. In section 4 we show for the AGQ family that the skewness measure proposed by Arnold & Groeneveld (1995) is a skewness measure in the sense of Oja. In section 6 we discuss some examples for the skewness functions $a(.)$ and $b(.)$.

2 Some concepts for the measurement of skewness

Oja (1981) p. 7 introduces a location-scale-skewness family of distributions as a family of distributions such that each pair of distributions is skewness comparable. This means that for each pair of distributions holds the van Zwet ordering of skewness.

Definition 1 *Let $\mathcal{F}$ be a family of cumulative distribution functions and $F, G \in \mathcal{F}$ and G^{-1} the quantile function of G.*

1. *F and G will be called skewness comparable if $G^{-1}(F(x))$ is either convex or concave on the support of F.*

2. *F is not more skewed to the right than G (shortly: $F \preceq_2 G$) if $G^{-1}(F(x))$ is convex on the support of F.*

3. *$\mathcal{F}$ is a locations-scale-skewness family if each pair $(F, G) \in \mathcal{F}$ is skewness comparable.*

Transformations of location and scale do not matter the fact that $G^{-1}(F(x))$ is convex or concave. This was proven by Klein & Fischer (2006). We summarize the result in the following lemma.

Lemma 1 *Let F, G be cumulative distribution functions with quantile function G^{-1}. Then*

$$G^{-1}(F(x)) \text{ convex (concave) on } supp(F) \Longrightarrow c + dG^{-1}\left(F\left(\tfrac{x-a}{b}\right)\right),\ b, d > 0,$$
$$a, c \in \mathbb{R} \text{ convex (concave) on } supp(F).$$

This lemma allows to discuss only distributions F and G with unimodal densities and common modus 0.

Whether $G^{-1}(F(.))$ is convex or concave can be checked by the sign of its second derivative. The following lemma proven by Klein & Fischer (2006), p. 1167 or stated by Arnold & Groeneveld (1995), p. 35 gives a sufficient and necessary condition for the convexity (concavity) of $G^{-1}(F(.))$.

Lemma 2 *Let F, G be continuous, cumulative distribution functions with densities f and g. f and g shall be differentiable on $\mathbb{R}$. Define $\phi_f(x) = -f'(x)/f^2(x)$, $\phi_g(x) = -g'(x)/g(x)^2$, $x \in \mathbb{R}$. F^{-1} and G^{-1} are the quantile functions corresponding to F and G. Then $G^{-1}(F(x))$ is convex (concave) on $\mathbb{R}$ iff*

$$\phi_f(F^{-1}(u)) - \phi_g(G^{-1}(u)) \leq (\geq) 0 \ \text{ for all } u, v \in (0, 1). \tag{3}$$

Skewness shall be measured by a functional that maps a set of distributions in the real numbers and satisfies some requirements that are plausible for the concept of skewness. Oja (1981) gives the following definition for a measure of skewness. If $F \in \mathcal{F}$ belongs to the random variable X, $a \times F + b$ denotes the distribution function of the transformed random variable $aX + b$, $a, b \in \mathbb{R}$.

Definition 2 *Let $\mathcal{F}$ be a family of distributions. $T : \mathcal{F} \to \mathbb{R}$ is a measure of skewness in $\mathcal{F}$ if*

1. *$T(a \times F + b) = sgn(a)T(F)$ for all $a, b \in \mathbb{R}$, $F \in \mathcal{F}$.*
2. *$T(F) \leq T(G)$ if $F, G \in \mathcal{F}$ and $F \preceq_2 G$.*

As a consequence of this definition for a measure of skewness holds

$$T((-1) \times F) = -T(F) \tag{4}$$

This means that reflection of the distribution changes the sign of the measure of skewness.

From the large class of possible measure of skewness we will discuss the proposal of Arnold & Groeneveld (1995) in more detail. If Y is a random variable with uniquely defined modus y_M they propose

$$AG = P(Y < y_M) - P(Y \geq y_M) = 1 - 2P(Y < y_M)$$

as a measure of skewness. AG takes values in $[-1, 1]$. As Ferreira & Steel (2006) p. 823 pronounce this measure "is fairly intuitive for unimodal distributions with negative (positive) values for left (right) skewed distributions and 0 for symmetric distributions".

To show that the AGQ family is a location-scale-skewness family in the sense of Oja the functions $G^{-1}(F(x))$ and ϕ_f have to be computed for this special family.

3 Some functions for the AGQ family

Let F denote the cumulative distribution function of a random variable X and assume that F is continuous on $\mathbb{R}$ and has a density f which itself is differentiable on $\mathbb{R} \setminus \{0\}$. Further, we assume that X is symmetrically distributed. Without restriction of generality we assume that the median of X is 0. Otherwise, we consider $Y = X - \text{median}(X)$. F^{-1} denotes the quantile function of X. We also assume that the support of F is $\mathbb{R}$. The results of the paper can be easily transferred to the case of a random variable X with compact support.

Let X_γ be the random variable with density (1). Then it is easy to verify that X_γ has the following cumulative distribution function

$$\begin{aligned} F(x;\gamma) = & \frac{2a(\gamma)}{a(\gamma)+b(\gamma)} F\left(\frac{x}{a(\gamma)}\right) I(x<0) \\ & + \frac{a(\gamma)-b(\gamma)}{a(\gamma)+b(\gamma)} + \frac{2b(\gamma)}{a(\gamma)+b(\gamma)} F\left(\frac{x}{b(\gamma)}\right) I(x \geq 0). \end{aligned} \tag{5}$$

Inverting the cumulative distribution function leads to the quantile function of X_γ:

$$\begin{aligned} F^{-1}(u;\gamma) = & a(\gamma) F^{-1}\left(\frac{a(\gamma)+b(\gamma)}{2a(\gamma)} u\right) I(A) \\ & + b(\gamma) F^{-1}\left(\frac{a(\gamma)+b(\gamma)}{2b(\gamma)}\left(u - \frac{a(\gamma)-b(\gamma)}{a(\gamma)+b(\gamma)}\right)\right) I(\bar{A}) \end{aligned} \tag{6}$$

with $A = \{u \in [0,1] | u \leq a(\gamma)/(a(\gamma)+b(\gamma)\}$.

With this quantile function we derive the median of X_γ as

$$F^{-1}(0.5;\gamma) = \begin{cases} a(\gamma) F^{-1}\left(\frac{a(\gamma)+b(\gamma)}{a(\gamma)}\right) \leq 0 & \text{for } a(\gamma) > b(\gamma) \\ b(\gamma) F^{-1}\left(\frac{3b(\gamma)-a(\gamma)}{4b(\gamma)}\right) \geq 0 & \text{for } a(\gamma) < b(\gamma) \end{cases} \tag{7}$$

With the help of the cumulative and the inverse distribution functions we get (2). Γ denotes the domain of $a(.)$ and $b(.)$. To restrict the number of cases we have to discuss, we assume that $a(\gamma)$ is monotone increasing with existing derivative $a'(\gamma) > 0$ and $b(\gamma)$ is monotone decreasing with existing derivative $b'(\gamma) < 0$ for $\gamma \in \Gamma$. Then, $a(\gamma)/b(\gamma)$ is monotone increasing and $b(\gamma)/a(\gamma)$ monotone decreasing on Γ.

Let $\gamma_1, \gamma_2 \in \Gamma$ with $\gamma_2 < \gamma_1$ be fixed. Then we get after some tedious calculations

$$\Lambda(x;\gamma_1,\gamma_2) = \begin{cases} a(\gamma_2) F^{-1}\left(\frac{1+b(\gamma_2)/a(\gamma_2)}{1+b(\gamma_1)/a(\gamma_1)} F\left(\frac{x}{a(\gamma_1)}\right)\right) \\ \quad \text{for } x \leq a(\gamma_1) F^{-1}\left(\frac{1}{2}\frac{1+b(\gamma_1)/a(\gamma_1)}{1+b(\gamma_2)/a(\gamma_2)}\right) \\ b(\gamma_2) F^{-1}\left(\frac{1+a(\gamma_2)/b(\gamma_2)}{1+b(\gamma_1)/a(\gamma_1)} F\left(\frac{x}{a(\gamma_1)}\right) - \frac{a(\gamma_2)-b(\gamma_2)}{2b(\gamma_2)}\right) \\ \quad \text{for } a(\gamma_1) F^{-1}\left(\frac{1}{2}\frac{1+b(\gamma_1)/a(\gamma_1)}{1+b(\gamma_2)/a(\gamma_2)}\right) < x \leq 0 \\ b(\gamma_2) F^{-1}\left(\frac{1+a(\gamma_2)/b(\gamma_2)}{1+a(\gamma_1)/b(\gamma_1)} \frac{a(\gamma_1)-b(\gamma_1)}{2b(\gamma_1)} - \frac{a(\gamma_2)-a(\gamma_2)}{2b(\gamma_2)} + \frac{1+a(\gamma_2)/b(\gamma_2)}{1+a(\gamma_1)/b(\gamma_1)} F\left(\frac{x}{b(\gamma_1)}\right)\right) \\ \quad \text{for } x > 0 \end{cases} \tag{8}$$

Whether $\Lambda(.;\gamma_1,\gamma_2)$ is convex or concave can be checked with the following ϕ-function: Let $\phi(x) = -f'(x)/f^2(x)$ and $\phi(x;\gamma) = -f'(x;\gamma)/f^2(x;\gamma)$ for $x \in \mathbb{R}$. Then it is easy to show that

$$\begin{aligned}\phi(x;\gamma) = {} & \frac{1}{2}\left(1+\frac{b(\gamma)}{a(\gamma)}\right)\phi\left(\frac{x}{a(\gamma)}\right)I(x<0) \\ & +\frac{1}{2}\left(1+\frac{a(\gamma)}{b(\gamma)}\right)\phi\left(\frac{x}{b(\gamma)}\right)I(x\geq 0).\end{aligned} \tag{9}$$

Let $I \subseteq supp(F(.;\gamma_1)$ such that $f'(x;\lambda_1)$ and $f'(F^{-1}(F(x;\lambda_1);\lambda_2);\lambda_2)$ exists for $x \in I$ then $\Lambda(.;\gamma_1,\gamma_2)$ is convex (concave) on I iff

$$\phi(x;\gamma_1) - \phi\left(F^{-1}\left(F(x;\gamma_1);\gamma_2\right);\gamma_2\right) \leq (\geq) 0$$

for $x \in I$. Obviously, the interval I may not contain $x = 0$ or $x = F^{-1}(a(\gamma_2)/(a(\gamma_2) + b(\gamma_2));\gamma_1)$.

An important property is the monotonicity of ϕ to show under which conditions $\Lambda(.;\gamma_1,\gamma_2)$ is convex or concave. Klein & Fischer (2006) discuss some examples for distributions such that $\phi(.)$ is monotone increasing. To these distributions belong among others the Gaussian and the t distribution, the Laplace distribution and the generalized secant hyperbolic distribution of Vaughan (2002). A counterexample is the generalized t distribution of McDonald & Newey (1988).

4 AGQ family as a location-scale-skewness family

Let $\mathcal{F}$ be the AGQ family with densities (1). Then we have to show that $F(.;\gamma_1), F(.;\gamma_2))$ are skewness comparable. This means that $F^{-1}(F(x;\gamma_1);\gamma_2)$ is either convex or concave on $\mathbb{R}$. We will prove this result under some assumptions. The first set of assumptions concerns the symmetric density f. Especially, $\phi(x) = -f'(x)/f(x)^2$ has to be monotone increasing with derivative $\phi'(x) > 0$. The second set of assumptions concerns the skewness functions $a(.)$ and $b(.)$. $a(.)$ and $b(.)$ shall be differentiable on Γ with $a'(\gamma) > 0$ and $b'(\gamma) < 0$ for $\gamma \in \Gamma$. This means that $a(\gamma)/b(\gamma)$ is monotone increasing on the common domain Γ.

Notice that the ϕ-function of $F(.;\gamma)$ is only defined for $x \neq 0$ because $f(.;\gamma)$ is continuous, but not differentiable at $x = 0$. This demands a special treatment at $x = 0$.

Two lemmata will prepare the main result that the AGQ family is a location-scale-skewness family in the sense of Oja.

Lemma 3 *Let F be a continuous distribution function with unimodal and symmetric density function f that is continuous on $\mathbb{R}$ and differentiable for $\{\mathbb{R}\setminus 0\}$ such that $\phi'(x) > 0$ for $x \neq 0$. Denote $\mathcal{F} = \{F(.;\gamma)|\gamma \in \Gamma\}$ the AGQ family of distributions with positive skewness functions $a(.)$ and $b(.)$. Furthermore, we assume that $a(.)$ an d b(.) are differentiable with $a'(\gamma) > 0$ and $b'(\gamma) < 0$ for $\gamma \in \Gamma$. Then*

$$\frac{\partial\phi(F^{-1}(u;\gamma);\gamma)}{\partial\gamma} = \frac{\partial\phi(x;\gamma)}{\partial\gamma}|_{x=F^{-1}(u;\gamma)}\frac{\partial F^{-1}(u;\gamma)}{\partial\gamma} < 0$$

for $u < a(\gamma)/a(\gamma) + b(\gamma))$ *or* $u > a(\gamma)/a(\gamma) + b(\gamma))$ *and* $u \in (0, 1)$.

Proof:

- Discussion of $\partial\phi(x;\gamma)/\partial\gamma$ for $x \neq 0$:

  Let $x < 0$:

$$\frac{\partial\phi(x;\gamma)}{\partial\gamma} = 1/2\frac{\partial b(\gamma)/a(\gamma)}{\partial\gamma}\phi\left(\frac{x}{a(\gamma)}\right) +1/2(1 + b(\gamma)/a(\gamma))\phi'\left(\frac{x}{a(\gamma)}\right)\left(-\frac{x}{a(\gamma)^2}\right)a'(\gamma) > 0$$

  because $a'(\gamma) > 0$, $b'(\gamma) < 0$, $\phi(x) < 0$ for $x < 0$ and $\phi'(x) > 0$ for $x \in \mathbb{R}$.

  Let $x > 0$. Then we get

$$\frac{\partial\phi(x;\gamma)}{\partial\gamma} = 1/2\frac{\partial a(\gamma)/b(\gamma)}{\partial\gamma}\phi\left(\frac{x}{b(\gamma)}\right) +1/2(1 + a(\gamma)/b(\gamma))\phi'\left(\frac{x}{b(\gamma)}\right)\left(-\frac{x}{b(\gamma)^2}\right)b'(\gamma) > 0$$

  due to $\phi(x) > 0$, $x > 0$.

- Discussion of $\partial F^{-1}(.;\gamma)/\partial\gamma$ for $u \neq a(\gamma)/(a(\gamma) + b(\gamma)$:

  Moreover, fur $u < a(\gamma)/(a(\gamma) + b(\gamma))$

$$\frac{\partial F^{-1}(u;\gamma)}{\partial\gamma} = a'(\gamma)F^{-1}\left(1/2\left(1 + b(\gamma)/a(\gamma)\right)u\right) +1/2a(\gamma)\frac{1}{f\left(F^{-1}\left(1/2(1 + b(\gamma)/a(\gamma)\right)u\right)}\frac{\partial b(\gamma)/a(\gamma)}{\partial\gamma} < 0$$

  because $F^{-1}\left(1/2\left(1 + b(\gamma)/a(\gamma)\right)u\right) < 0$ for $u < a(\gamma)/(a(\gamma) + b(\gamma))$ and $\frac{\partial b(\gamma)/a(\gamma)}{\partial\gamma} < 0$.

  Fur $u > a(\gamma)/(a(\gamma) + b(\gamma))$ we get

$$\frac{\partial F^{-1}(u;\gamma)}{\partial\gamma} = b'(\gamma)F^{-1}\left(1/2\left(1 + a(\gamma)/b(\gamma)\right)u + 1 - a(\gamma)/b(\gamma)\right) +1/2b(\gamma)\frac{1}{f\left(F^{-1}\left(1/2(1 + b(\gamma)/a(\gamma)\right)u + 1 - a(\gamma)/b(\gamma)\right)} \cdot\frac{\partial a(\gamma)/b(\gamma)}{\partial\gamma}(u - 1) < 0$$

  because $F^{-1}\left(1/2\left(1 + a(\gamma)/b(\gamma)\right)u + 1 - a(\gamma)/b(\gamma)\right) > 0$ for $u > a(\gamma)/(a(\gamma) + b(\gamma))$, $\frac{\partial b(\gamma)/a(\gamma)}{\partial\gamma} > 0$ and $u - 1 < 0$.

- Combining these results we get

$$\frac{\partial \phi(F^{-1}(u;\gamma);\gamma)}{\partial \gamma} = \frac{\partial \phi(x;\gamma)}{\partial \gamma}|_{x=F^{-1}(u;\gamma)} \frac{\partial F^{-1}(u;\gamma)}{\partial \gamma} < 0$$

for $u < a(\gamma)/a(\gamma) + b(\gamma))$ or $u > a(\gamma)/(a(\gamma) + b(\gamma)$ and $u \in (0,1)$. $\square$

Lemma 4 *Let F be a continuous distribution function with unimodal and symmetric density function f that is positive and continuous on $\mathbb{R}$. Let $F(.;\gamma_1)$, $F(.;\gamma_2)$ be elements of the AGQ family of distributions. Then*

$$\Lambda'(x;\gamma_1,\gamma_2) = \frac{\partial F^{-1}(F(x;\gamma_1);\gamma_2)}{\partial x} > 0 \text{ for } x \in \mathbb{R}$$

and continuous on $\mathbb{R}$.

Proof: It holds

$$\Lambda'(x;\gamma_1,\gamma_2) = \frac{f(x;\gamma_1)}{f(F^{-1}(F(x;\gamma_1);\gamma_2);\gamma_2)} \quad x \in \mathbb{R}. \tag{10}$$

$f(.;\gamma) > 0$ and $F(.;\gamma)$ are continuous functions on $\mathbb{R}$. $F^{-1}(u;\gamma)$ is continuous on $(0,1)$. Therefore, $\Lambda'(.;\gamma_1,\gamma_2) > 0$ and continuous for $x \in \mathbb{R}$. $\square$

Theorem 1 *Let F be a continuous distribution function with unimodal and symmetric density function f that is positive and continuous on $\mathbb{R}$ and differentiable for $\{\mathbb{R}\backslash 0\}$ such that $\phi'(x) > 0$ for $x \neq 0$. Denote $\mathcal{F} = \{F(.;\gamma)|\gamma \in G\}$ the AGQ family of distributions skewness functions $a(.)$ and $b(.)$. Furthermore, we assume that $a(.)$ and b(.) are differentiable with $a'(\gamma) > 0$ and $b'(\gamma) < 0$ for $\gamma \in \Gamma$. Then $\mathcal{F}$ is a location-scale-skewness family.*

Proof: We have to show that all members of $\mathcal{F}$ are skewness comparable. This means that $\Lambda(x;\gamma_1,\gamma_2)$ is either convex or concave on $\mathbb{R}$ for all $\gamma_1,\gamma_2 \in \Gamma$.

With lemma 3 we know that $\phi(F^{-1}(u;\gamma)$ is monotone decreasing in γ for $u \neq a(\gamma)/(a(\gamma)+b(\gamma)$.

For $\gamma_2 < \gamma_1$ it is

$$\frac{a(\gamma_2)}{a(\gamma_2)+b(\gamma_2)} < \frac{a(\gamma_1)}{a(\gamma_1)+b(\gamma_1)}.$$

The interval $(0,1)$ can be divided into three subsets:

$$I_1 = \left(0, \frac{a(\gamma_2)}{a(\gamma_2)+b(\gamma_2)}\right), \quad I_2 = \left(\frac{a(\gamma_2)}{a(\gamma_2)+b(\gamma_2)}, \frac{a(\gamma_1)}{a(\gamma_1)+b(\gamma_1)}\right), \quad I_3 = \left(\frac{a(\gamma_1)}{a(\gamma_1)+b(\gamma_1)}\right).$$

$\phi(F^{-1}(u;\gamma))$ is monotone decreasing in γ for $u \in I_1 \cup I_2 \cup I_3$). This means that

$$\phi(F^{-1}(u;\gamma_1),\gamma_1) - \phi(F^{-1}(u;\gamma_2);\gamma_2) \text{ for } u \in I_1 \cup I_2 \cup I_3.$$

Therefore, $\Lambda(.;\gamma_1,\gamma_2)$ is convex on the intervals

$$J_1 = \left(-\infty, F^{-1}\left(\frac{a(\gamma_2)}{(a(\gamma_2)+b(\gamma_2)}\right)\right), \quad J_2 = \left(F^{-1}\left(\frac{a(\gamma_2)}{(a(\gamma_2)+b(\gamma_2)}, 0\right)\right), \quad J_3 = (0,\infty)$$

corresponding to I_1, I_2 and I_3. With lemma 4 $\Lambda'(.;\gamma_1,\gamma_2)$ is continuous on $\mathbb{R}$. Therefore, we can generalize the convexity of $\Lambda(.;\gamma_1,\gamma_2)$ from the three subsets J_1, J_2, J_3 to the whole real line.

A similar discussion for $\gamma_2 > \gamma_1$ leads to the conclusion that $\Lambda(.;\gamma_1,\gamma_2)$ is concave on $\mathbb{R}$ for this case. $\Box$.

Notice that $\phi(F^{-1}(u;\gamma))$ is strictly decreasing. Therefore, $\Lambda(.;\gamma_1,\gamma_2)$ is strictly increasing and either strictly convex or strictly concave on $\mathbb{R}$.

5 Skewness measure of Arnold & Groeneveld for the AGQ family

Arnold & Groeneveld (1995) propose

$$AG = P(Y < y_M) - P(Y > y_M) = 1 - 2P(Y < y_M).$$

as a measure of skewness for a random variable Y with unique modus y_M. For a symmetric density $f(.)$ with modus 0 $f(,;\gamma)$ also has modus 0 with $F(0;\gamma) = a(\gamma)/(a(\gamma)+b(\gamma)$. Therefore, we get for the skewness measure of Arnold & Groeneveld

$$AG(F(.;\gamma)) = 1 - 2\frac{a(\gamma)}{a(\gamma)+b(\gamma)} = \frac{b(\gamma)-a(\gamma)}{a(\gamma)+b(\gamma)}.$$

Notice that AG does not depend on the underlying symmetric density $f(.)$. Under the assumption that $a(.)$ (($b(.)$) is monotone increasing (decreasing) AG is monotone decreasing in γ. This means

$$\gamma_2 < \gamma_1 \iff AG(\gamma_1) < AG(\gamma_2).$$

Now, we want to show for the AGQ family that AG is a measure of skewness in the sense of Oja.

Corollary 5.1 *Let F be a continuous distribution function with unimodal and symmetric density function f that is continuous on $\mathbb{R}$ and differentiable for $\{\mathbb{R} \setminus 0\}$ such that $\phi'(x) > 0$ for $x \neq 0$. Denote $\mathcal{F} = \{F(.;\gamma)|\gamma \in G\}$ the AGQ family of distributions skewness functions $a(.)$ and $b(.)$. Further, we assume that $a(.)$ and b(.) are differentiable with $a'(\gamma) > 0$ and $b'(\gamma) < 0$ for $\gamma \in \Gamma$. Then AG is measure of skewness in the sense of Oja.*

Proof:

1. Let X_γ be distributed with $F(x;\gamma)$, $\gamma \in \Gamma$. X_γ has modus 0 such that $(X_\gamma - b)/a$ has modus $-b/a$.

   Consider $a \times F(.;\gamma) + b$ for $a, b \in \mathbb{R}$. Then

$$AG(a \times F(.;\gamma) + b) = 1 - 2P\left(\frac{X_\gamma - b}{a} < -\frac{b}{a}\right).$$

We get

$$AG(a \times F(.;\gamma) + b) = \begin{cases} 1 - 2P(X_\gamma < 0) = AG(F(.;\gamma) & \text{for } a > 0 \\ 1 - 2P(X_\gamma > 0) & \\ = 1 - 2(1 - P(X_\gamma < 0)) & \\ = -1 + 2P(X_\gamma < 0) = -AG(F(.;\gamma) & \text{for } a < 0 \end{cases}$$

Hence, $AG(a \times F(.;\gamma) + b) = sgn(a)AG(F(.;\gamma)$.

2. It remains to show that if $F(.;\gamma_1) \preceq_2 F(.;\gamma_2)$ then it holds $AG(\gamma_1) \le AG(\gamma_2)$.

   From lemma 3 and the proof of theorem 1 we know that $\phi(F^{-1}(.;\gamma)$ is a strictly decreasing function in γ for $u \in I_1 \cup I_2 \cup I_3$ if $\Lambda(.;\gamma_1,\gamma_2)$ is strictly convex on $\mathbb{R}$. Therefore, it holds for $\gamma_1, \gamma_2 \in \Gamma$:

$$\phi(F^{-1}(.;\gamma_1)) - \phi(F^{-1}(.;\gamma_2) \Longrightarrow \gamma_2 < \gamma_1.$$

   $AG(.)$ is strictly decreasing on Γ. This leads to

$$\Lambda(.;\gamma_1,\gamma_2) \text{ is srictly convex on } \mathbb{R} \Longrightarrow AG(F(.;\gamma_1)) < AG(F(.;\gamma_2)).$$

   for $\gamma_1, \gamma_2 \in \Gamma$. □

6 Special cases

6.1 Symmetry for $\gamma = 0$ and invariance w.r.t to reflection

For monotone increasing (decreasing) functions $a(.)$ ($b(.)$) the density (1) is symmetric around 0 if $a(\gamma) = b(\gamma)$.

1. For $a(\gamma) = \gamma$ and $b(\gamma) = 1/\gamma$ we get symmetry for $\gamma = 1$.
2. Setting $a(\gamma) = 1 + \gamma$ and $b(\gamma) = 1 - \gamma$ gives symmetry for $\gamma = 0$.

After the reparametrization $a(\gamma) = e^\gamma$ and $b(\gamma) = e^{-\gamma}$ in the first case we get also symmetry for $\gamma = 0$. Therefore, we restrict us on functions $a(.)$ and $b(.)$ such that $a(\gamma) = b(\gamma)$ for $\gamma = 0$. This means that 0 is in the domain Γ.

Let X_γ the random variable corresponding to the density (1). $-X_\gamma$ denotes the reflected random variable with reflected properties of skewness. If X_γ is skewed to the right, $-X_\gamma$ should be skewed to the left in the same manner. It is plausible to require that the skewness of $-X_\gamma$ should be the same as the skewness of $X_{-\gamma}$. This leads to the condition

$$f(x;-\gamma) = f(-x;\gamma) \;\; x \in \mathbb{R}$$

for $\gamma \in \Gamma$. For $x = 0$ we get

$$f(0;-\gamma) = \frac{2}{a(-\gamma) + b(-\gamma)} f(0) = \frac{2}{a(\gamma) + b(\gamma)} f(0) = f(0;\gamma).$$

This condition is satisfied if

$$b(\gamma) = a(-\gamma) \ \ \gamma \in \Gamma. \tag{11}$$

The splitting of scale originally proposed by Fechner gives an example for this special case. Another example is $a(\gamma) = e^{\gamma}$ and $b(\gamma) = e^{-\gamma}$ for $\gamma \in \mathbb{R}$.

Under (11) Arnold & Groeneveld's measure of skewness AG takes the form

$$AG(\gamma) = \frac{a(-\gamma) - a(\gamma)}{a(\gamma) + a(-\gamma)}.$$

6.2 Asymmetric distributions due to Fechner

We follow Cassart et al. (2008) and discuss a slightly more general version of Fechner's normal density with different scale parameters for positive and negative arguments. Set

$$a(\gamma) = 1 + a\gamma \ \text{ and } \ b(\gamma) = 1 - b\gamma, \ \ \gamma \in (-1, 1) \tag{12}$$

with $a, b > 0$. Obviously, $a(\gamma)$ is monotone increasing and $b(\gamma)$ is monotone decreasing for $\gamma > 0$. This proofs the following corollary.

Corollary 6.1 *Let F be a continuous distribution function with unimodal and symmetric density function f that is continuous on $\mathbb{R}$ and differentiable for $\{\mathbb{R} \setminus 0\}$ such that $\phi'(x) > 0$ for $x \neq 0$. Consider the special AGQ family of skewed densities*

$$f(x;\gamma) = \frac{2}{2 + (a-b)\gamma}\left(f\left(\frac{x}{1+a\gamma}\right) I(x \le 0) + f\left(\frac{1}{1-b\gamma}\right) I(x > 0)\right) \ \ x \in \mathbb{R}$$

with $a, b > 0$ and $\gamma \in (-1, 1)$. Then, this family is location-scale-skewness family in the sense of Oja.

Under the setting (12) we get for the Arnold & Groeneveld's measure of skewness

$$AG(F(.;\gamma)) = -\frac{(a+b)\gamma}{2 + (a-b)\gamma}.$$

In the Fechner case with $a = b = 1$ we get $AG(F.;\gamma) = -\gamma$ for $\gamma \in [-1, 1]$. This means that Arnold & Groeneveld's skewness measure and the negative of the skewness parameter γ are identical.

6.3 Asymmetric distribution due to Fernández et al.

We consider the proposal of Fernández et al. for skewed distributions with the modified choice

$$a(\gamma) = e^{\gamma} \ \text{ and } \ b(\gamma) = e^{-\gamma} \ \ \gamma \in \mathbb{R}. \tag{13}$$

$a(.)$ is monotone increasing and $b(.)$ monotone decreasing on $\mathbb{R}$ and $a(\gamma) > b(\gamma)$ for $\gamma > 0$. This proves the following corollary which was already proven by Klein & Fischer (2006).

Corollary 6.2 *Let F be a continuous distribution function with unimodal and symmetric density function f that is continuous on $\mathbb{R}$ and differentiable for $\{\mathbb{R} \setminus 0\}$ such that $\phi'(x) > 0$ for $x \neq 0$. Consider the special AGQ family of skewed densities*

$$f(x;\gamma) = \frac{2}{e^{\gamma} + e^{-\gamma}} \left(f\left(xe^{-\gamma}\right) I(x \leq 0) + f\left(xe^{\gamma}\right) I(x > 0) \right) \quad x \in \mathbb{R}$$

with $\gamma \in \mathbb{R}$. Then, this family is location-scale-skewness family in the sense of Oja.

The calculation of Arnold & Groeneveld's AG is straightforward.

7 Summary

There are several procedures to construct a skewed distribution. One of these procedures splits the value of a parameter of scale for the two sides of a symmetric distribution. We show that the most general form of this technique of generating skewed distributions proposed by Arellano-Valle et al. (2005) incorporates a well-defined parameter of skewness. It is well-defined in the sense that the generated distributions are skewed to the right if the parameter of skewness takes values less than 1. As second property we show that the parameter of skewness is compatible with the ordering $\preceq_2$ of van Zwet (1964) which is the strongest ordering in the hierarchy of orderings discussed by Oja (1981). In this sense the generated skewed distributions can be ordered by the parameter of skewness. We show how the measure of skewness of Arnold & Groeneveld depends on the skewness parameter.

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