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Abstract

There are several procedures to construct a skewed distribution. One of these procedures is based on a symmetric distribution that will be distorted by a skewed distribution defined on $(0, 1)$. This proposal stems from Arellano-Valle et al. and was refined by Ferreira & Steel. Up to now, it is an open question whether the famous skewness ordering of van Zwet will be preserved for this proposal. There is a general condition under which the van Zwets skewness ordering will be preserved by the Ferreira-Steel family. But this condition is not easy to verify for the most families of distribution. Therefore, for the skewness mechanism we choose a special beta distribution with only one parameter. Then, we get three results. First, the skewness ordering will be preserved starting for symmetric distributions that are leptokurtic like the logistic distribution. Larger parameter values give distributions that are more skewed to the right. Second, the same skewness mechanism can generate distributions that are more skewed to left if the support of the underlying symmetric distribution is compact. Third, for underlying symmetric distributions on $\mathbb{R}$ with platykurtic behavior the van Zwet ordering of skewness will be preserved. This restricts a little bit the benefit of the Ferreira-Steel family.

Keywords: Skewness; skewness to the right; skewness ordering, measure of skewness

1 Introduction

Starting with a symmetric density f with corresponding distribution function F Ferreira & Steel (2006) proposes

$$f(x; p) = f(x)p[F(x)], \quad x \in I_F \tag{1}$$

as a class of skewed distributions where the skewness comes from the skewed density p defined on $[0, 1]$. This proposal belongs to the large class of methods that introduce skewness into an originally symmetric distribution. To this methods count, for example, the hidden truncation model (see f.e. Azzalini (1985) or Arnold & Beaver (2002)), inverse scale factors in positive negative orthants (see Fernández & Steel (1998)) and order statistics (Jones (2004)).

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In all these cases the skewness will be introduced in an intuitive and plausible manner. Van Zwet (1964) and later Oja (1981) propose and discuss formal properties a family of skewed distributions should have. Van Zwet (1964) introduced the skewness ordering. This ordering means that a distributions F is not more skewed to the right than a distribution G if $G^{-1}(F(x))$ is convex for all $x \in I_F$ with the support I_F of F. Klein & Fischer (2006) show for a family of skewed distributions generated by splitting a scale parameter in the negative and positive orthants that this family preserves the Zwet ordering of skewness. They have to claim mild assumptions on the underlying symmetric distribution. Klein (2011) generalizes this result to the more general family of skewed distributions introduced by Arellano-Valle et al. (2005).

In the following we want to discuss the van Zwet ordering of skewness for the Ferreira-Steel family, too. But, for this family it will be seen that we cannot give a general proof. The corresponding sufficient condition is not easy to check for the most distributions f and p. Therefore, we concentrate on a very simple skewness mechanism by the choice of a beta distribution with one parameter a for p. If f is the density of a leptokurtic distribution (like the logistic distribution) we can show that the induced Ferreira-Steel family is a well defined family of skewed distributions. In contrast to this result, for rather platykurtic symmetric densities (like the power exponential distribution with parameter $\beta > 2$) the corresponding Ferreira-Steel family does not preserve the van Zwet ordering. This restricts the usefulness of this skewness mechanism. In the case of a symmetric distribution with compact support (like the uniform distribution on $(-1/2, 1/2)$ we can see that the elements of the Ferreira-Steel family are ordered the other way round. This means that higher values of a give distributions that are more skewed to left.

The paper is organized as follows. We give a short introduction in some concepts of skewness measuring. Then, the Ferreira-Steel family will be discussed. For this family sufficient condition will be derived for the van Zwet ordering. After this, we restrict the discussion to the special skewness mechanism generated by a beta distribution with one parameter. In this case the sufficient conditions look a little bit easier. But, they still depend heavily on the comparison of score functions and hazard rates. Finally, we further restrict the discussion on three types of symmetric distributions. For each of this types we get different results with respect to the van Zwet ordering.

2 Some concepts for the measurement of skewness

Oja (1981) p. 7 introduces a location-scale-skewness family of distributions as a family of distributions such that each pair of distributions is skewness comparable. This means that for each pair of distributions the van Zwet (or convex) ordering of skewness holds.

Definition 1 *Let $\mathcal{F}$ be a family of cumulative distribution functions and $F, G \in \mathcal{F}$ and G^{-1} the quantile function of G.*

1. *F and G will be called skewness comparable if $G^{-1}(F(x))$ is either convex or concave on the support of F.*

2. *F is not more skewed to the right than G (shortly: $F \preceq_2 G$) if $G^{-1}(F(x))$ is convex on the support of F.*

3. *$\mathcal{F}$ is a locations-scale-skewness family if each pair $(F, G) \in \mathcal{F}$ is skewness comparable.*

Whether $G^{-1}(F(.))$ is convex or concave can be checked by the sign of its second derivative. The following lemma proven by Klein & Fischer (2006), p. 1167 or stated by Arnold & Groeneveld (1995), p. 35 gives a sufficient and necessary condition for the convexity (concavity) of $G^{-1}(F(.))$.

For distributions F and G such that the corresponding densities f and g are differentiable we give a sufficient condition for the convex ordering.

Lemma 1 *Let F, G be continuous, cumulative distribution functions with densities f and g. f and g shall be differentiable on I_F and I_G. Define $\phi_f(x) = -f'(x)/f^2(x)$, $\phi_g(x) = -g'(x)/g(x)^2$, $x \in I_G$. F^{-1} and G^{-1} are the quantile functions corresponding to F and G. Then $G^{-1}(F(x))$ is convex (concave) on $\mathbb{R}$ iff*

$$\phi_f(F^{-1}(u)) - \phi_g(G^{-1}(u)) \leq (\geq) 0 \text{ for all } u \in (0,1). \tag{2}$$

The so-called convex ordering $\preceq_2$ was introduced by van Zwet. This ordering has no reference to any measure of location or scale. MacGillivray (1986), p. 997 stated that "any weakening of the ordering in the sense of covering larger classes of distributions, involves reference to particular location and scale parameters". MacGillivray discusses five different ordering and proves their interrelationships. The weakest ordering she considered is

$$F \preceq_5 G :\Longleftrightarrow \frac{G^{-1}(1-u) - G^{-1}(1/2)}{G^{-1}(u) - G^{-1}(1/2)} \leq \frac{F^{-1}(1-u) - F^{-1}(1/2)}{F^{-1}(u) - F^{-1}(1/2)} \quad u \in (0,1). \tag{3}$$

The notation $\preceq_5$ stems from Arnold & Groeneveld (1993). Both authors argue that orderings based on skewness functionals are preferable to those related to convex orderings. We do not agree with this statement. Following MacGillivray (1986) p. 997 a ordering has to be identified without reference to a preferred skewness functional. After this identification only such skewness functionals should be used that preserves this ordering.

Skewness shall be measured by a functional that maps a set of distributions in the real numbers and satisfies some requirements that are plausible for the concept of skewness. Oja (1981) gives the following definition for a measure of skewness. If $F \in \mathcal{F}$ belongs to the random variable X, $a \times F + b$ denotes the distribution function of the transformed random variable $aX + b$, $a, b \in \mathbb{R}$.

Definition 2 *Let $\mathcal{F}$ be a family of distributions. $T : \mathcal{F} \to \mathbb{R}$ is a measure of skewness in $\mathcal{F}$ if*

1. *$T(a \times F + b) = sgn(a)T(F)$ for all $a, b \in \mathbb{R}$, $F \in \mathcal{F}$.*

2. *$T(F) \leq T(G)$ if $F, G \in \mathcal{F}$ and $F \preceq_2 G$.*

As a consequence of this definition for a measure of skewness holds

$$T((-1) \times F) = -T(F) \tag{4}$$

This means that reflection of the distribution changes the sign of the measure of skewness.

From the large class of possible measures of skewness we will discuss the proposal of Arnold & Groeneveld (1995) in more detail. If Y is a random variable with uniquely defined modus y_M they propose

$$AG = P(Y < y_M) - P(Y \geq y_M) = 1 - 2P(Y < y_M)$$

as a measure of skewness. AG takes values in $[-1, 1]$. As Ferreira & Steel (2006) p. 823 pronounce this measure "is fairly intuitive for unimodal distributions with negative (positive) values for left (right) skewed distributions and 0 for symmetric distributions".

To show that the Ferreira Steel-family is a location-scale-skewness family in the sense of Oja the functions $G^{-1}(F(x))$ and ϕ_f have to be calculated.

3 Some functions for the Ferreira-Steel family

Let F denote the cumulative distribution function of a random variable X and assume that F is continuous on $\mathbb{R}$ and has a density f which itself is differentiable on $\mathbb{R}$. Furthermore, we assume that X is symmetrically distributed. Without restriction of generality we assume that the median of X is 0. Otherwise, we consider $Y = X - \text{median}(X)$. F^{-1} denotes the quantile function of X.

Let X_p be the random variable with density (1). Then it is easy to verify that X_p has the following cumulative distribution function

$$F(x; p) = P[F(x)] \;\; x \in \mathbb{R} \tag{5}$$

with the cumulative distribution function P. This leads to the quantile function of X_p

$$F^{-1}(u; p) = F^{-1}[P^{-1}(u)], \;\; u \in (0, 1). \tag{6}$$

With the help of the cumulative and the inverse distribution functions we get (**??**).

$$\Lambda(x; p_1, p_2) = F^{-1}[P_2^{-1}[P_1[F(x)]]], \;\; x \in I_F. \tag{7}$$

Whether $\Lambda(.; p_1, p_2)$ is convex or concave on I_F can be checked with the following ϕ-function: Let $\phi_f(x) = -f'(x)/f^2(x)$, $\phi_p(u) = -p'(u)/p(u)^2$ and $\phi(x; p) = -f'(x; p)/f^2(x; p)$ for $x \in I_F$, $p \in (0, 1)$. Then it is easy to show that

$$\phi(x; p) = \phi_f(x) \frac{1}{p[F(x)]} + \phi_p[F(x)], \;\; x \in I_F. \tag{8}$$

Due to (2) $\Lambda(.; p_1, p_2)$ is convex (concave) on I_F if and only if

$$\phi(F^{-1}[P_1^{-1}(u); p_1] - \phi(F^{-1}[P_2^{-1}(u); p_2] \leq (\geq) 0$$

for $u \in (0,1)$. Inserting (8) this condition becomes

$$\frac{\phi_f(F^{-1}[P_1^{-1}(u)])}{p_1[P_1^{-1}(u)]} - \frac{\phi_f(F^{-1}[P_2^{-1}(u)]}{p_2[P_2^{-1}(u)]} + \phi_{p_1}(P_1^{-1}(u)) - \phi_{p_2}(P_2^{-1}(u)) \leq (\geq) 0 \qquad (9)$$

for $u \in (0,1)$. If P_1 is not more skewed the right that P_2 than the second difference of (9) is non positive. If $P_1^{-1}(u) \leq P_2^{-1}(u)$, $0 < u < 1$ it could be helpful that ϕ_f is increasing on I_F. Klein & Fischer (2006) discuss some examples for distributions such that $\phi_f(.)$ is increasing. To these distributions belong among others the Gaussian and the t distribution, the Laplace distribution and the generalized secant hyperbolic distribution of Vaughan (2002). A counterexample is the generalized t distribution of McDonald & Newey (1988). But we cannot conclude from an increasing function ϕ_f that the first difference in (9) is also non positive. This depends on the values of $p_i[P_i^{-1}(u)]$ for $u \in (0,1)$ and $i = 1, 2$.

Ferreira & Steel (2006) specify a special skewing mechanism p that preserves the unimodality and the tail behavior of the underlying distribution F. Startíng with the function h as twice the distribution function of a symmetric β-distribution on $[0, 1/2]$

$$h(x) = 4\int_0^x \frac{1}{\beta(d+1, d+1)} u^d (1-u)^d du, \quad x \in [0, 1/2]$$

with $d \in \mathbb{N}_0$ they define the function

$$g(x) = h\left(\frac{e^{\delta x} - 1}{2(e^{\delta/2} - 1)}\right) I_{(0,1/2]}(x) + \left(2 - h\left(\frac{e^{\delta(x-1/2)} - 1}{2(e^{\delta/2} - 1)}\right)\right) I_{(1/2,1]}(x).$$

The density of the skewing mechanism p is given by

$$p(x) \propto \left(1 + \frac{M-1}{g(1/2)}\right) I_{(0,1/2]}(x) + \left(r + \frac{M-r}{g(1/2)} g(x)\right) I_{(1/2,1]}(x).$$

$M > \max\{1, r\}$ determines the modal value and r the tail ratio. If $d \in \mathbb{N}_0$ h has a polynomial form. This choice has the advantage that the corresponding distribution function P can be derived in an analytical form. But, even in the simplest case $d = 0$ with $h(x) = 4x$, $x \in [0, 1/2]$ it is not possible to check analytically whether the convex ordering will be preserved. This is due to fact that the quantile function P^{-1} can only be calculated numerically. Even the score function $-p'/p$ has a rather complicated form.

For this reason and to simplify the discussion we concentrate on distributions P_i, $i = 1, 2$ with an explicit representation of the corresponding quantile function. This is the case if we consider beta distributions with one parameter a_i, $i = 1, 2$ with monotone increasing densities on $[0, 1]$. These special beta distributions are extremely skewed to the left.

4 Skewness of one parametric beta distributions

Lemma 2 *Consider the distribution functions P_1 and P_2 with*

$$P_i(u) = u^{a_i}, \quad u \in [0,1], \ a_i > 0, \ i = 1, 2.$$

Then, P_1 not more skewed to the right than P_2 if and only if $a_1 \geq a_2$.

Proof: The result follows immediately from the fact that

$$\Lambda(u, a_1, a_2) = P_2^{-1}(P_1(u)) = u^{a_1/a_2}, \ \ 0 < u < 1, \ a_1, a_2 > 0.$$

is convex on $[0, 1]$ if and only if $a_1 \geq a_2$. $\Box$

Equivalently, we get the convexity of $\Lambda(., a_1, a_2)$ from

$$\phi_{p_1}(P_1^{-1}(u)) - \phi_{p_2}(P_2^{-1}(u)) = \frac{a_2 - a_1}{a_1 a_2} u^{-1} \leq \ \ 0 < u < 1, \ a_1, a_2 > 0$$

for $a_1 > a_2$, where p_i and P_i^{-1} are the corresponding density $p_i(u) = a_i u^{a_i - 1}$ and the quantile function $P_i^{-1}(u) = u^{1/a_i}$, $0 \in [0, 1]$, $a_i > 0$, $i = 1, 2$. For the ϕ-functions holds:

$$\phi_{p_i}(u) = -\frac{1 - a_i}{a_i} \frac{1}{u^{a_i}}, \ \ u \in [0, 1], \ a_i > 0.$$

In the sense of Oja the parameter a_i is a measure of skewness for this family of special beta distributions.

For $a > 1$, the modus of $p(u) = a u^{a-1}$, $u \in [0, 1]$ lies at $u = 1$. Therefore, $AG = -1$ for all $a > 1$. This indicates that for $a > 1$ all one parametric beta distributions are extremely skewed to the left.

Now, we want to investigate how these skewness properties of one parametric beta distribution will be transferred to the corresponding Ferreira-Steel family.

5 Skewness of the Ferreira-Steel family

Theorem 1 *Let $P_i(u) = u^{a_i}$, $u \in [0, 1]$, $a_i > 0$, $i = 1, 2$ and f, F, F^{-1} the density, distribution and quantile function of a random variable with support I_F distributed symmetrically around 0. The derivatives f' and f'' of f will be assumed to exist. $\psi_f(x) = -f'(x)/f(x)$, $x \in I_F$ denotes the score function. For $a_1 > a_2$, $\Lambda(x; a_1, a_2) = F^{-1}[P_2^{-1}[P_1[F(x)]]]$ is convex (concave) for $x \in I_F$ if*

$$\Delta(x) = \frac{f(x)}{F(x)} + \psi_f(x) + \left((\psi_f'(x) + \psi_f(x)^2) \frac{F(x)}{f(x)} + \psi_f(x) \right) \ln F(x) \geq (\leq) 0$$

for all $x \in I_F$.

Proof: For (9) we get with $x = F^{-1}(P_1^{-1}(u))$

$$\begin{aligned} D(x) &= \frac{\phi_f(x)}{p_1[F(x)]} - \frac{\phi_f\left(F^{-1}[P_2^{-1}[P_1[F(x)]]]\right)}{p_2\left(F^{-1}[P_2^{-1}[P_1[F(x)]]]\right)} + \phi_{p_1}(x) - \phi_{p_2}(P_2^{-1}[P_1[F(x)]]) \\ &= \frac{-f'(x)/f(x)}{f(x) a_1 F(x)^{a_1 - 1}} - \frac{-f'(F^{-1}\left(F(x)^{a_1/a_2}\right))/f(F^{-1}\left(F(x)^{a_1/a_2}\right))}{f\left(F^{-1}\left(F(x)^{a_1/a_2}\right)\right) a_2 F(x)^{a_1 - a_1/a_2}} \\ &+ \frac{a_2 - a_1}{a_1 a_2} \frac{1}{F(x)^{a_1}}, \ \ x \in I_F. \end{aligned}$$

Setting $a = a_1/a_2$ it follows

$$D(x) = \frac{F(x)^{-a_1}}{a_1}\left((1-a) - \frac{f'(x)}{f(x)}\frac{F(x)}{f(x)} + a\frac{f'\left(F^{-1}\left(F(x)^a\right)\right)}{f\left(F^{-1}\left(F(x)^a\right)\right)}\frac{F(x)^a}{f\left(F^{-1}\left(F(x)^a\right)\right)}\right), \quad x \in I_F.$$

The sign of $D(x)$ will be determined by the sign of

$$\lambda(x;1) - \lambda(x;a), \quad x \in I_F$$

with

$$\lambda(x;a) := a + a\frac{f'\left(F^{-1}(F(x)^a)\right)}{f\left(F^{-1}(F(x)^a)\right)}\frac{F(x)^a}{f\left(F^{-1}(F(x)^a)\right)}, \quad x \in I_F.$$

For $a_1 > a_2$, $\Lambda(x; a_1, a_2)$ is convex (concave) for $x \in I_F$ if $\lambda(x;a)$ is increasing (decreasing) in a for all $x \in I_F$. Due to the assumption that f is twice differentiable we can calculate the derivative of λ with respect to a as

$$\begin{aligned}\frac{\partial\lambda(x;a)}{\partial a} &= 1 + \phi(y)F(y) + a\left(\phi'(y)\frac{F(y)}{f(y)} + \phi(y)\right)F(y)\frac{1}{a}\ln F(y) \\ &= \frac{F(y)}{f(y)}\left(\frac{f(y)}{F(y)} + \psi_f(y) + \left((\psi_f'(y) + \psi_f(y)^2)\frac{F(y)}{f(y)} + \psi_f(y)\right)\ln F(y)\right)\end{aligned}$$

with $\psi_f(y) = -f'(y)/f(y)$, $\phi_f(y) = \psi(y)/f(y)$ and $y = F^{-1}(F(x)^a)$, $x \in I_F$. For $a > 1$, $\lambda(x;a)$ is increasing (decreasing) in a if

$$\Delta(y) = \frac{f(y)}{F(y)} + \psi_f(y) + \left(\psi_f'(y) + \psi_f(y)^2)\frac{F(y)}{f(y)} + \psi_f(y)\right)\ln F(y) \geq (\leq) 0$$

with $y = F^{-1}(F(x)^a)$, $x \in I_F$. $\square$

The function Δ will be determined by the hazard rate $f(y)/F(y)$ and the score function ψ_f. If $\Delta(y) \geq (\leq) 0$ for all y from the support of F, $\Lambda(.; a_1, a_2)$ is convex (concave) for $a_1 > a_2 > 0$.

We want to discuss these functions in several examples. These examples will show that the sign of $\Delta(y)$ can be negative for all $y \in \mathbb{R}$, can be positive for distributions with compact support and can be indeterminate if the underlying symmetric distribution has no compact support and is platykurtic.

6 Some examples

6.1 Concavity for the skewed logistic distribution

Corollary 6.1 *Let $P_i(u) = u^{a_i}$, $u \in (0,1)$, $a_1 > 0$ and F the distribution function of the logistic distribution*

$$F(x) = \frac{1}{1+e^{-x}}, \quad x \in \mathbb{R}.$$

$\Lambda(x; a_1, a_2) = F[P_2^{-1}[P_1[F(x)]]]$ *is convex for $x \in \mathbb{R}$ if $a_1 > a_2$.*

Proof: For this situation we have to discuss $\Delta(x)$. $\Delta(x)$ depends on

$$\begin{aligned}
f(x) &= F(x)(1-F(x)) = \frac{e^{-x}}{1+e^{-x}} \\
\frac{f(x)}{F(x)} &= 1 - F(x) = \frac{e^{-x}}{1+e^{-x}} \\
\psi(x) &= -\frac{d \ln f(x)}{dx} = 2F(x) - 1 \\
\psi'(x) &= 2f(x)
\end{aligned}$$

for $x \in \mathbb{R}$. Inserting these expressions in $\Delta(x)$ we get

$$\begin{aligned}
\Delta(x) &= F(x) + \frac{F(x)}{1-F(x)} \ln F(x) \\
&= F(x)\left(1 - \frac{1+e^{-x}}{e^{-x}} \ln\left(1+e^{-x}\right)\right) \\
&= F(x)\left(1 - \left(1+\frac{1}{e^x}\right) \ln\left(1+\frac{1}{e^x}\right)^{e^x}\right).
\end{aligned}$$

For $x \to \infty$ we get

$$1 + \frac{1}{e^x} \to 1 \text{ and } \left(1+\frac{1}{e^x}\right)^{e^x} \to e$$

such that

$$\Delta(x) \to 1 - 1 \ln e = 0.$$

To show $\Delta(x) \leq 0$ for $x \in \mathbb{R}$, we consider the derivative of

$$1 - \left(1+\frac{1}{e^x}\right) \ln\left(1+\frac{1}{e^x}\right)^{e^x} = 1 - \ln\left(1+\frac{1}{e^x}\right)^{e^x+1}$$

for $x \in \mathbb{R}$. If this derivative is non negative then we can conclude that $\Delta(x) \leq 0$ for $x \in \mathbb{R}$. Calculating the derivative gives

$$\begin{aligned}
&-\left(1+\frac{1}{e^x}\right)^{e^x+1}\left(e^x \ln\left(1+\frac{1}{e^x}\right) + (e^x+1)\frac{1}{1+1/e^x}\left(-\left(\frac{1}{e^x}\right)^2 e^x\right)\right) \\
&= -\left(1+\frac{1}{e^x}\right)^{e^x+1} e^x \left(\ln\left(1+\frac{1}{e^x}\right) - \frac{1}{e^x}\right).
\end{aligned}$$

Because $\ln(1+y) \leq y$ holds for $y > 0$ we get

$$-\left(1+\frac{1}{e^x}\right)^{e^x+1} e^x \left(\ln\left(1+\frac{1}{e^x}\right) - \frac{1}{e^x}\right) \geq 0 \text{ for } x \in \mathbb{R}.$$

Putting the results together we get $\Delta(x) \leq 0$ for $x \in \mathbb{R}$. $\square$

This means that starting with the logistic distribution with the density f and the distribution function F

$$\Lambda(x; p_1, p_2) = F^{-1}(P_2^{-1}(P_1(F(x)))) = x^{a_1/a_2}$$

is concave on $\mathbb{R}$ for $a_1 > a_2 > 0$.

The upper part of figure 1 gives an impression that the skewness mechanism generates densities which are skewed to the right. The function $\Lambda(x, a_1, a_2)$ in the lower part is concave for all $x \in \mathbb{R}$.

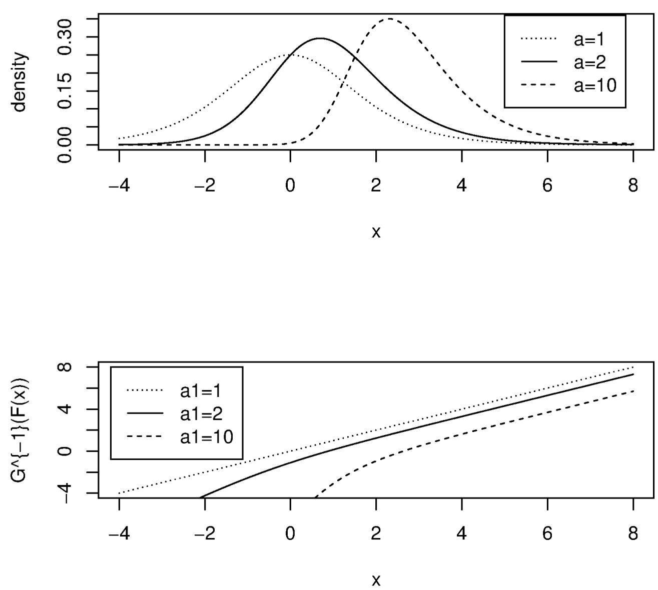

Figure 1: Skewed logistic densities for $a = 1, 2, 10$ and $\Lambda(.; a_1, a_2)$ for $a_1 = 2, 10$ and $a_2 = 1$.

We want to calculate the skewness measure proposed by Arnold & Groeneveld for the skewed logistic distribution with density

$$f(x; a) = a\frac{e^{-x}}{(1 + e^{-x})^{a+1}}, \quad x \in \mathbb{R}, \; a > 0.$$

The mode of this distribution is given by $x = \ln a$. Inserting the mode in the distribution function $F(x; a)$ gives

$$F(\ln a; a) = \left(\frac{1}{1 + 1/a}\right)^a.$$

Then, we get

$$AG = 1 - 2F(\ln a; a) = 1 - 2\left(\frac{a}{a + 1}\right)^a$$

for $a > 0$. AG is a monotone increasing function of the skewness parameter a.

6.2 Convexity and compact support

Based on the logistic distribution the skewness mechanism of Ferreira & Steel gives a distribution that is more skewed to the right for higher parameter values a. We can also get skewed distributions that are as more skewed to the left if a increases as the example of an uniform distribution shows.

Corollary 6.2 *Let $P_i(u) = u^{a_i}$, $u \in [0,1]$, $a_i > 0$ and F the distribution function of the uniform distribution*

$$F(x) = x, \quad x \in [-1/2, 1/2].$$

$\Lambda(x; a_1, a_2) = F[P_2^{-1}[P_1[F(x)]]]$ *is convex for* $x \in [-1/2, 1/2]$ *if* $a_1 > a_2$.

Proof: The

$$\Lambda(x; p_1, p_2) = F^{-1}(P_2^{-1}(P_1(F(x)))) = x^{a_1/a_2}$$

is convex on $[-1/2, 1/2]$ for $a_1 > a_2 > 0$. $\square$

The upper part of figure 2 show densities that are extremely skewed to the left for $a = 2, 10$. This property stems the fact that the skewness mechanism p is already skewed to the left. The lower part shows the convexity of $\Lambda(x; a_1, a_2)$ for all $x \in [-1/2, 1/2]$.

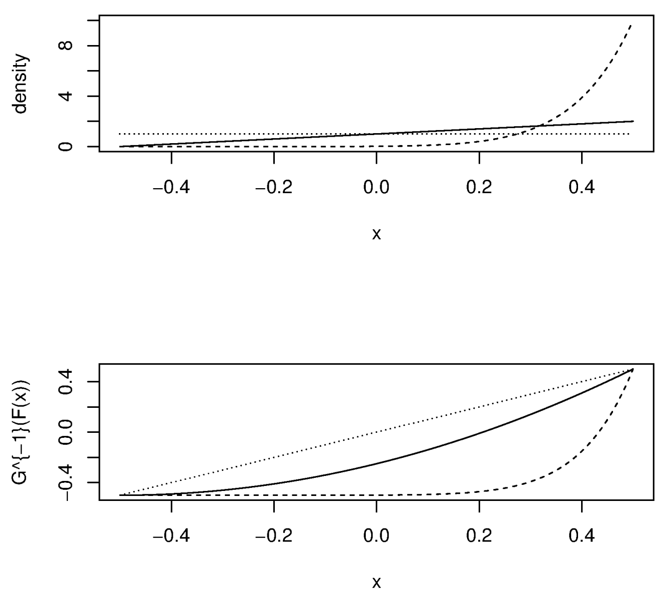

Figure 2: Skewed uniform densities for $a = 1, 2, 10$ and $\Lambda(.; a_1, a_2)$ for $a_1 = 2, 10$ and $a_2 = 1$.

6.3 Neither convex nor concave

6.3.1 Skewed power exponential distribution

For $a > 1$, it has to be that

$$F(x)^a \leq F(x) \;\; x \in \mathbb{R}. \tag{10}$$

This implies that

$$\Lambda(x; p_1, p_2) = F^{-1}(P_2^{-1}(P_1(F(x)))) = F^{-1}(F(x)^{a_1/a_2}) \leq x \;\; x \in \mathbb{R}. \tag{11}$$

If $0 < F(x) < 1$ (i.e. no compact support) for $x \in \mathbb{R}$ and $\psi(x^*) = \psi'(x^*) = 0$ for some $x^* \in \mathbb{R}$, then

$$\Delta(x^*) = \frac{f(x^*)}{F(x^*)} > 0.$$

Therefore, $F^{-1}(P_2^{-1}(P_1(F(.))))$ can neither be concave or convex on $\mathbb{R}$. Convexity means that there is point where $F^{-1}(P_2^{-1}(P_1(F(.))))$ has to cut the identity line. This contradicts (11).

Example 6.1 *As an example for a distribution with the properties $0 < F(x < 1$, $x \in \mathbb{R}$ and $\psi(x^*) = psi'(x^*)$ for some x^* we consider the power exponential distribution with parameter $\beta > 2$.*

Consider the density

$$f(x) = \frac{\beta}{2^{1/\beta+1}} \Gamma\left(\frac{1}{\beta}\right) e^{-1/2|x|^\beta} \;\; x \in \mathbb{R}, \; \beta > 0.$$

The density is symmetric around 0. Distribution and quantile function can be easily derived from the gamma distribution.

The score function and the corresponding derivative are

$$\psi(x) = 1/2sign(x)\beta|x|^{\beta-1}, \;\; and \;\; \psi'(x) = 1/2\beta(\beta-1)|x|^{\beta-2} \tag{12}$$

for $x \in \mathbb{R}$. For $\beta > 2$ it is $\psi(0) = \psi'(0) = 0$ and

$$\Delta(0) = 2\frac{\beta}{2^{1/\beta+1}} \Gamma\left(\frac{1}{\beta}\right) > 0.$$

hold. For $\beta > 2$, this means that the skewed exponential distributions $f(x; a) = f(x)F(x)^a$, $x \in \mathbb{R}$ cannot be ordered with respect to the skewness ordering of van Zwet.

The upper part of figure 1 shows the densities of the skewed power exponential distribution with exponent $\beta = 3$ and the parameter values $a = 1$ (=symmetry), $a = 2$ and $a = 10$. Even this extreme setting with $a = 10$ does not produce a skewed power exponential distribution with a visible amount of skewness in any direction. This corresponds with the conclusion we can draw from the lower part of the figure. Here, we can see explicitly that $\Lambda(x; a_1, a_2)$ is neither convex nor concave for all $x \in \mathbb{R}$. We consider the skewness parameter values $a = a_1/a_2 = 2$ and $a = a_1/a_2 = 10$.

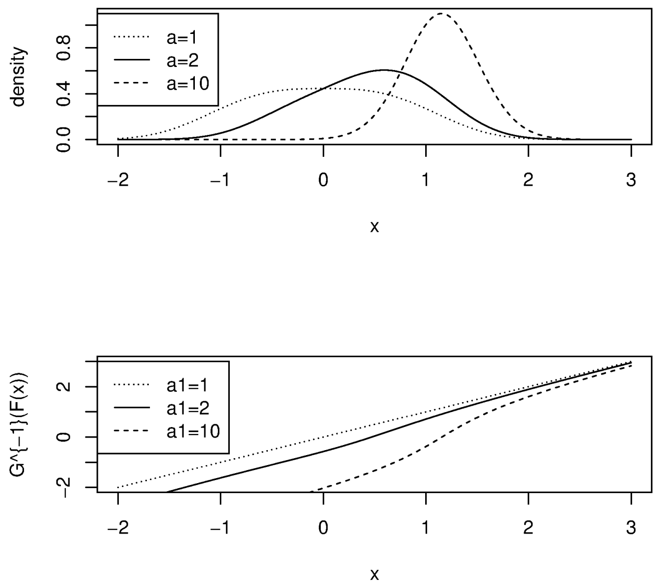

Figure 3: Skewed power exponential density for $\beta = 3$ and $a = 2, 3, 4$ and $\Lambda(.; a_1, a_2)$ for $a_2 = 1$ and $a_1 = 2, 10$.

6.3.2 Weaker ordering of skewness

Now, we want to investigate whether the family of skewed power exponential distributions cant be ordered by a ordering that is weaker than the convex ordering. MacGillivray (1986) discussed the ordering (3). Figure 4 shows the graph of the function

$$Z(u; a) = \frac{F^{-1}((1-u)^{1/a}) - F^{-1}((1/2)^{1/a})}{F^{-1}(u^{1/a}) - F^{-1}((1/2)^{1/a})}, \quad u \in [0, 1]$$

for the values $a = a_1 = 3$ and $a = a_2 = 2$. We can see that there is a cutting point between the two curves. This implies that the weak ordering $\preceq_5$ will not be preserved by the family of skewed power exponential distributions.

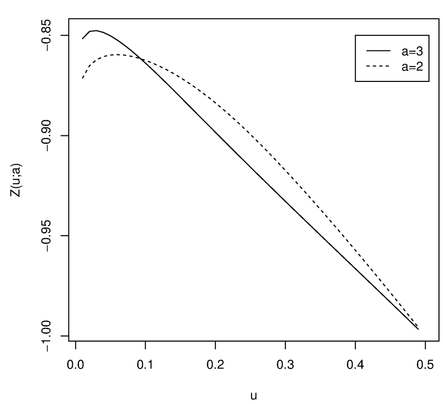

Figure 4: $Z(.; a)$ for $\beta = 3$ and $a = 2, 3$.

6.3.3 Fechner asymmetry for the power exponential distribution

Now, we want to answer the question whether asymmetry can be introduced into the family of power exponential distributions by an other skewness mechanism. We consider the general approach of Arellano-Valle et al. (2005). Starting with a symmetric density f they introduce a asymmetric distribution by

$$f(x;\gamma) = \frac{2}{a(\gamma)+b(\gamma)}\left(f\left(\frac{x}{a(\gamma)}\right)I(x<0) + f\left(\frac{x}{b(\gamma)}\right)I(x\geq 0)\right) \tag{13}$$

where $a(.)$, $b(.)$ are known positive functions with domain G. This is a general kind of generating skewness by splitting a scale parameter for the negative and positive half of a distribution that includes several well-known special cases. The idea of splitting scale parameter values can be traced back to the publications of Fechner (see f.e. Fechner (1897)).

Klein (2011) shows under which conditions this Arellano-Valle family preserves the van Zwet ordering of skewness.

Theorem 2 *Let F be a continuous distribution function with unimodal and symmetric density function f that is continuous on $\mathbb{R}$ and differentiable for $\{\mathbb{R}\setminus 0\}$ such that $\phi_f'(x) > 0$ for $x \neq 0$. Further, we assume that $a(.)$ and b(.) are differentiable with $a'(\gamma) > 0$ and $b'(\gamma) < 0$ for $\gamma \in G$. If $\gamma_2 < \gamma_1$,*

$$\Lambda(x;\gamma_1,\gamma_2) = F^{-1}(F(x;\gamma_1);\gamma_2) \ \text{ is convex on } \mathbb{R}.$$

All we have to do is to show that the derivative of $\phi_f(x) = -f'(x)/f(x)^2$ is positive for $x \neq 0$ if f is the density of a power exponential distribution.

Example 6.2 *From (12) we know that*

$$\phi'(x) = \frac{1}{f(x)}\left(\psi'(x)+\psi(x)^2\right) = \frac{1}{f(x)}\left(\frac{1}{2}\beta(\beta-1)|x|^{\beta-2} + \frac{1}{4}\beta^2|x|^{\beta-1}\right),\ x \in \mathbb{R}.$$

This expression is positive for $x \neq 0$.

7 Summary

There are several procedures to construct a skewed distribution. One of these procedures is based on a symmetric distribution that will be distorted by a skewed distribution defined on $[0,1]$. This proposal stems from Arellano-Valle et al. and was refined by Ferreira & Steel. Up to now, it is an open question whether the famous skewness ordering of van Zwet will be preserved for this proposal. There is a general condition under which the van Zwets skewness ordering will be preserved by the Ferreira-Steel family. But this condition is not easy to verify for the most families of distribution. Therefore, for the skewness mechanism we choose a special beta distribution with only one parameter. This corresponds with an extremely negative skewness. Then, we get three results. First, the skewness ordering will be preserved starting for symmetric distributions that are leptokurtic like the logistic distribution. Larger parameter values give distributions that are more skewed to the right. Second, the same skewness mechanism can generate distributions that are more skewed to left if the support of the underlying symmetric distribution is compact. Third, for underlying symmetric distributions on $\mathbb{R}$ with platykurtic behavior the van Zwet ordering of skewness will be preserved. This restricts the benefit of the Ferreira-Steel family. In this case, the alternative of Arellano-Valle et al. (2005) to generate skewness by choosing different values for the scale parameter in left and right orthant of the distribution seems to be more successful.

Up to now, there is no proof that in the case of the normal distribution the proposal of Ferreira & Steel generates skewed distributions which can preserve the van Zwet ordering. This is a real lack because this case is the most important one for applications.

References

[1] Arellano-Valle, R.B., H.W. Gómez, F.A. Quintana: *Statistical Inference for a General Class of Asymmetric Distributions*. Journal of Statistical Planning and Inference, **128**:427-443, 2005.

[2] Arnold, B.C. and Beaver, R.J.: *Skewed Multivariate Models Related to Hidden Truncation and/or Selective Reporting*. Test, **11**:7-54, 2002

[3] Arnold, B.C. and R.A. Groeneveld: *Skewness and Kurtosis Orderings: An Introduction*. Stochastic Inequalities. IMS Lecture Notes - Monograph Series, **22**:17-24, 1993.

[4] Arnold, B.C. and R.A. Groeneveld: *Measuring Skewness with Respect to the Mode*. The American Statistician, **49**(1):34-38, 1995.

[5] Azzalini, A.: *A class of distributions which includes the normal ones*. Scandinavian Journal of Statistics, **12**:171-178, 1985.

[6] Fechner, G.T.: *Kollektivmaßlehre*. Engelmann, Leipzig, 1897.

[7] Fernández, C. and Steel, M.F.J.: *On Bayesian Modeling of Fat Tails and Skewness*, Journal of the American StatisticalAssociation, **93**:359-371, 1998.

[8] Ferreira, T.A.S. and M.F.J. Steel: *A Constructive Representation of Univariate Skewed Distributions*. Journal of the American Statistical Association, **101**:823-829, 2006.

[9] Jones, M.C.: *Families of Distributions Arising From Distributions of Order Statistics*, Test **13**:1-43, 2004.

[10] Klein, I., M. Fischer: *Skewness by Splitting the Scale Parameter*. Communications in Statistics, Theory and Methods, **35**(7):1159-1171, 2006.

[11] Klein, I.: *Van Zwet Ordering for Fechner Asymmetry*. IWQW-Discussion Papers, University of Erlangen-Nürnberg 8/2011.

[12] McDonald, J. B. and W. K. Newey: *Partially Adaptive Estimation of Regression Models via the Generalized-t-Distribution*. Econometric Theory, **1**(4):428-457, 1988.

[13] MacGillivray, H.L.: *Skewness and Asymmetry: Measures and Orderings*. Annals of Statistics, **14**: 994-1011, 1986.

[14] Oja, H.: *On Location, Scale, Skewness and Kurtosis of Univariate Distributions*. Scandinavian Journal of Statistics, **18**:154-168, 1981.

[15] Van Zwet, W. R.: *Convex Transformations of Random Variables*. Mathematical Centre Tracts No. 7. Mathematical Centre, Amsterdam, 1964.

[16] Vaughan, D. C.: *The Generalized Hyperbolic Secant distribution and its Application*. Communications in Statistics – Theory and Methods, **31**(2):219-238, 2002.

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